PDE Approaches for the Long-Time Behavior of Diffusion Processes

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This note is dedicated to the long-time behavior of diffusion processes via PDE approaches.

1 Theory of hypocoercivity

1.1 From Langevin dynamics to $L = A^*A + B$

Let $\mathcal{H}$ be a separable Hilbert space. The theory of hypocoercivity [1] is to study diffusion processes with generator in the form

$$L = A^*A + B, \tag{1.1}$$

where A, B are two unbounded operator in $\mathcal{H}$ and B is anti-symmetric, i.e., $B = -B^*$. For such an operator, its kernel $\mathcal{K}$ (a subset of $\mathcal{H}$) is characterized by $\mathcal{K} = \ker(A) \cap \ker(B)$. Also, an important property of the operator L is the semigroup e^{-tL} is a contraction, i.e., $\|e^{-tL}\| \leqslant 1$, $\forall t \geqslant 0$.

Usually, the Hilbert space $\mathcal{H}$ is chosen to be $L^2(\mu_\infty)$, where μ_∞ is the equilirbium of a given diffusion process. Also, $A = (A_1, \cdots, A_m)$ and B are usually in the form

$$A_j h = a_j \cdot \nabla h, \quad Bh = b \cdot \nabla h.$$

Here, h is usually the probability density with respect to μ_∞. Let ρ_∞ be the probability density of μ_∞ with respect to the Lebesgue measure. If we compactly write $Ah = \sigma \nabla h$, where σ is an $m \times n$ matrix, then we have

1. $B = -B^* \Longleftrightarrow \nabla \cdot (b\rho) = 0$;
2. $A^* g = -\nabla \cdot (\sigma^* g) - \langle \nabla \rho_\infty, \sigma^* g \rangle$.

Therefore, for a given diffusion process, we may try to derive the operator $L = A^*A + B$ in the following steps:

1. Find the unique invariant distribution μ_∞.
2. Rewrite the Fokker–Planck equation in terms of the probability density with respect to μ_∞.
3. Determine the operaror A according to the second-order part.
4. Determine the operator B according to the remaining part.

Now we present two examples of diffusion processes and derive the operator L in the $A^*A + B$ form.

1.1.1 Overdamped Langevin dynamics

The overdampled Langevin dynamics in $\mathbb{R}^n$ is given by

$$\mathrm{d}X_t = \sqrt{2}\sigma(X_t)\mathrm{d}W_t + \xi(X_t)\mathrm{d}t, \tag{1.2}$$

where $\xi \in C^1(\mathbb{R}^n;\mathbb{R}^n)$ is the drift function, and $\sigma \in C^2(\mathbb{R}^n;\mathbb{R}^{n\times m})$ is the diffusion coefficient, and W_t is the standard Wiener process in $\mathbb{R}^m$. The corresponding Fokker–Planck equation is

$$\frac{\partial \rho}{\partial t} = \nabla\cdot(D\nabla\rho - \xi\rho), \qquad D = \sigma^*\sigma. \tag{1.3}$$

Suppose the process admits a unique invariant distribution $\mu_\infty(\mathrm{d}x) = \rho_\infty(x)\mathrm{d}x$ (determined by ξ and σ), then the probability density $h(t,x) = \rho(t,x)/\rho_\infty$ satisfies the diffusion equation

$$\frac{\partial h}{\partial t} = \nabla\cdot(D\nabla h) - \Big(\xi - 2D\nabla\log\rho_\infty\Big)\cdot\nabla h. \tag{1.4}$$

Hence the operator L in $L^2(\mu_\infty)$ is given by $L = A^*A + B$, where the operators A, B are given by

$$\boxed{A = \sigma\nabla, \qquad B = \big(\xi - 2D\nabla\log\rho_\infty\big)\cdot\nabla.} \tag{1.5}$$

When the diffusion matrix D is non-degenerate, the kernel space $\mathcal{K}$ consists of constant functions.

1.1.2 Underdamped Langevin dynamics

The underdamped Langevin dynamics in $\mathbb{R}^n$ is given by

$$\begin{cases} \mathrm{d}X_t = V_t, \\ \mathrm{d}V_t = -V_t - \nabla V(X_t) + \sqrt{2}\mathrm{d}W_t, \end{cases} \tag{1.6}$$

where $V \in C^1(\mathbb{R}^n;\mathbb{R})$ is the potential function. When $V(x)$ satisifies the confinement condition, the unique invariant distribution μ_∞ is given by the density function

$$\rho_\infty(x,v) = \frac{1}{Z}\exp\Big(-V(x) - \frac{|v|^2}{2}\Big). \tag{1.7}$$

The Fokker–Planck equation is

$$\frac{\partial \rho}{\partial t} + v\cdot\nabla_x\rho - \nabla V(x)\cdot\nabla_v\rho = \Delta_v\rho + \nabla_v\cdot(vf). \tag{1.8}$$

The probability density $h(t,x,v) = \rho(t,x,v)/\rho_\infty$ then satisfies

$$\frac{\partial h}{\partial t} + v\cdot\nabla_x h - \nabla V(x)\cdot\nabla_v h = \Delta_v h - v\cdot\nabla_v h. \tag{1.9}$$

Hence the operator L in $L^2(\mu_\infty)$ is given by $L = A^*A + B$, where the operators A, B are given by

$$\boxed{A = \nabla_v, \qquad B = v\cdot\nabla_x - \nabla V(x)\cdot\nabla_v.} \tag{1.10}$$

The kernel space $\mathcal{K}$ consists of constant functions.

1.2 Hypocoercivity theorem

1.2.1 Basic theorem

We present the basic theorem of the hypocoercivity. It is based on the assumption that both A and A^* commute with $[A, B]$.

Theorem 1.1 *Consider the linear operator $L = A^*A + B$ where B is anti-symmetric. Let $C = [A, B]$ be the commutator. If there exist constant α, β such that*

1. *A and A^* commute with C;*
2. *$[A, A^*]$ is α-bounded to I and A;*
3. *$[B, C]$ is β-bounded to A, A^2, C and AC.*

Then there is a scalar product $(\!(\cdot,\cdot)\!)$ on $\mathcal{H}^1/\mathcal{K}$ such that

$$(\!(h, Lh)\!) \geqslant K(\|Ah\|^2 + \|Ch\|^2) \tag{1.11}$$

for some $K > 0$ which only depends on α and β.

Here, the norm in the Hilbert space $\mathcal{H}^1$ is defined by

$$\|h\|^2_{\mathcal{H}^1} = \|h\|^2 + \|Ah\|^2 + \|Ch\|^2, \tag{1.12}$$

which is the counterpart of the Sobolev space.

Proof In the proof, the scalar product $(\!(\cdot,\cdot)\!)$ is explicitly chosen as

$$(\!(h, g)\!) = \langle h, g\rangle + a\langle Ah, Ag\rangle + b\Re\langle Ah, Cg\rangle + b\Re\langle Ag, Ch\rangle + c\langle Ch, Cg\rangle. \tag{1.13}$$

In particular, when $h = g$,

$$(\!(h, h)\!) = \|h\|^2 + a\|Ah\|^2 + 2b\Re\langle Ah, Ch\rangle + c\|Ch\|^2. \tag{1.14}$$

Note that the commuation between A, C allows the norm defined above makes sense. This norm is equivalent to $\mathcal{H}^1$. The rate of change for the $(\!(\cdot,\cdot)\!)$ norm of $e^{-tL}h$ is

$$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\!(e^{-tL}h, e^{-tL}h)\!) = \Re(\!(e^{-tL}h, e^{-tL}Lh)\!). \tag{1.15}$$

By direct calculation, we have the estimate

$$\begin{aligned}
\Re(\!(h, Lh)\!) \geqslant{}& \|Ah\|^2 \\
&+ a\Big(\|A^2h\|^2 - \|Ah\|\|[A, A^*]Ah\| - \|Ah\|\|Ch\|\Big) \\
&+ \Big(\|Ch\|^2 - \|Ah\|\|R_2h\| - 2\|A^2h\|\|CAh\| - \|Ch\|\|[A, A^*]Ah\|\Big) \\
&+ \Big(\|CAh\|^2 - \|Ch\|\|R_2h\|\Big).
\end{aligned}$$

As a consequence, we have

$$\Re(\!(h, Lh)\!) \geqslant \langle X, mX\rangle_{\mathbb{R}^4}, \tag{1.16}$$

where X is the vector $(\|Ah\|, \|A^2h\|, \|Ch\|, \|CAh\|)$, and the 4×4 matrix m is given by

$$m = \begin{bmatrix} 1-(a\alpha+b\beta) & -(a\alpha+b\beta) & -(a+b\alpha+b\beta+c\beta) & -b\beta \\ 0 & a & -(b\alpha+c\beta) & -2b \\ 0 & 0 & b-c\beta & -c\beta \\ 0 & 0 & 0 & c \end{bmatrix}. \tag{1.17}$$

With appropriate choices of the parameters a, b, c, one may ensure that the symmetric part of m is positive definite, and hence the theorem holds true. □

We present a simple example that the conditions of the basic theorem hold true. Consider the underdamped Langevin dynamics, where the operators A, B are given by

$$A = \nabla_v, \qquad B = v\cdot\nabla_x - \nabla V(x)\cdot\nabla_v. \tag{1.18}$$

Recall that the density of the invariant distribution is

$$\rho_\infty(x,v) = \frac{1}{Z}\exp\Big(-V(x) - \frac{|v|^2}{2}\Big). \tag{1.19}$$

Then the adjoint operator A^* satisfies $A^*h = -\nabla_v(\rho_\infty h)$, and we have

$$\begin{aligned} ABh &= \nabla_v\big(v\cdot\nabla_x h - \nabla V(x)\cdot\nabla_v h\big) \\ &= \nabla_x h + v\cdot\nabla_{xh}^2 h - \nabla V(x)\cdot\nabla_v^2 h, \\ BAh &= v\cdot\nabla_x(\nabla_v h) - \nabla V(x)\cdot\nabla_v(\nabla_v h) \\ &= v\cdot\nabla_{xh}^2 h - \nabla V(x)\cdot\nabla_v^2 h. \end{aligned}$$

Therefore, the commutator C is explicitly given by $C = AB - BA = \nabla_x$, which commutes with A and A^*. Therefore, the basic theorem can be applied to the underdamped Langevin dynamics.

In this case, the kernel space $\mathcal{K}$ consists of constant functions, and norm in the Hilbert space $\mathcal{H}^1$ is defined by

$$\|h\|_{\mathcal{H}^1}^2 = \|h\|^2 + \|\nabla_x h\|^2 + \|\nabla_v h\|^2. \tag{1.20}$$

Therefore, $\mathcal{H}^1$ is the usual H^1 space corresponding to the probability measure μ_∞, while $\mathcal{H}$ itself is the usual L^2 space. Then we have the hypocoercivity

$$\|e^{-tL}\|_{\mathcal{H}^1/\mathcal{K}} \leqslant Ce^{-\lambda t} \tag{1.21}$$

for some constants $C, \lambda > 0$.

1.2.2 Generalized theorem

The generalized theorem of hypocoercivity allows a sequence of commutators.

Theorem 1.2 *Let $\mathcal{H}$ be a Hilbert space, and $A : \mathcal{H} \to \mathcal{H}^n$ and $B : \mathcal{H} \to \mathcal{H}$ be unbounded operators with $B^* = -B$. Let $L = A^*A + B$ and $\mathcal{K} := \ker L$. Assume there exists $N_c \in \mathbb{N}$ and operators $\{C_j\}_{j=0}^{N_c+1}$ and $\{R_j\}_{j=0}^{N_c+1}$ such that*

$$C_0 = A, \qquad [C_j, B] = Z_{j+1}C_{j+1} + R_{j+1}, \qquad C_{N_c+1} = 0. \tag{1.22}$$

And for $k \in \{0, 1, \cdots, N_c\}$,

1. $[A, C_k]$ *is bounded relatively to* $\{C_j\}_{j=0}^{k}$ *and* $\{C_j A\}_{j=0}^{k-1}$*;*
2. $[C_k, A^*]$ *is bounded relatively to* I *and* $\{C_j\}_{j=0}^{k}$*;*
3. R_k *is bounded relatively to* $\{C_j\}_{j=0}^{k-1}$ *and* $\{C_j A\}_{j=0}^{k-1}$*;*
4. *There are positive constants* λ_j, Λ_j *such that* $\lambda_j I \leqslant Z_j \leqslant \Lambda_j$*.*

Then there is a scalar product $(\!(\cdot,\cdot)\!)$ *on* $\mathcal{H}^1$*, which defines a norm equivalent to the* $\mathcal{H}^1$ *norm*

$$\|h\|_{\mathcal{H}^1}^2 = \|h\|^2 + \sum_{j=0}^{N_c} \|C_j h\|^2 \tag{1.23}$$

such that

$$\Re(\!(h, Lh)\!) \geqslant K \sum_{j=0}^{N_c} \|C_j h\|^2, \qquad \forall h \in \mathcal{H}^1/\mathcal{K}. \tag{1.24}$$

We note that each C_j consists of the high-order terms of A and B. After some commutator operations, the high-order terms vanish in C_{N_c+1}.

The proof is accomplished by defining the norm

$$(\!(h, h)\!) = \|h\|^2 + \sum_{k=0}^{N_c} \left(a_k \|C_k h\|^2 + 2\Re b_k \langle C_k h, C_{k+1} h\rangle \right) \tag{1.25}$$

and choosing the constants a_k and b_k carefully.

1.2.3 Entropic convergence theorem

The entropic convergence theorem requires the operators A, B to be exactly the first-order differential operators.

Theorem 1.3 *Let* $E \in C^2(\mathbb{R}^N)$ *such that* e^{-E} *is rapidly decreasing and* $\mu(X) = e^{-E(X)}\mathrm{d}X$ *is a probability measure on* $\mathbb{R}^N$*. Let* $(A_j)_{1\leqslant j\leqslant m}$ *and* B *be first-order derivation operators with smooth coefficients. Denote* A_j^* *and* B^* *by their respective adjoints in* $L^2(\mu)$*, and assume* $B = -B^*$*. Denote* A *by the collection* $(A_1, \cdots, A_m)$ *and define*

$$L = A^*A + B = \sum_{j=1}^{m} A_j^* A_j + B. \tag{1.26}$$

Assume there exists $N \in \mathbb{N}_c$*, derivation operators* $C_0, C_1, \cdots, C_{N_c+1}$*,* $R_1, \cdots, R_{N_c+1}$ *and vector-valued functions* $Z_1, \cdots, Z_{N_c+1}$ *such that*

$$C_0 = A, \qquad [C_j, B] = Z_{j+1}C_{j+1} + R_{j+1}, \qquad (0 \leqslant j \leqslant N_c), \qquad C_{N_c+1} = 0, \tag{1.27}$$

and

1. $[A, C_k]$ *is pointwise bounded to* A*;*
2. $[C_k, A^*]$ *is pointwise bounded to* $I, \{C_j\}_{0\leqslant j\leqslant k}$*;*
3. R_k *is pointwise bounded to* $\{C_j\}_{0\leqslant j\leqslant k-1}$*;*

4. *There are postive constants* λ_j, Λ_j *such that* $\lambda_j \leqslant Z_j \leqslant \Lambda_j$*;*
5. $[A, C_k]^*$ *is pointwise bounded to* I, A*.*

Then there exists a quadratic form S *in* $\mathbb{R}^N$ *such that the functional*

$$\mathcal{E}(h) = \int h \log h \mathrm{d}\mu + \int \frac{\langle S\nabla h, \nabla h\rangle}{h} \mathrm{d}\mu \tag{1.28}$$

satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(e^{-tL}h) \leqslant -\alpha \int \frac{\langle S\nabla h, \nabla h\rangle}{h} \mathrm{d}\mu. \tag{1.29}$$

In the proof, the functional $\mathcal{E}$ is chosen to be the form

$$\mathcal{E}(h) = \int fu + \sum_{k=0}^{N_c} \left(a_k \int f|C_k u|^2 + 2b_k \int f\langle C_k u, C_{k+1}u\rangle \right), \tag{1.30}$$

where a_k, b_k are the parameters to be determined.

2 Generalized Γ calculus for the long-time behavior

The generalized Γ calculus was developed in [2] to study the long-time behavior of a wide class of stochastic processes. It extends the classical Bakry–Emery theory and applies to the degenerate diffusion processes.

2.1 Generalized Γ calculus

Consider a stochastic process in $\mathbb{R}^d$ with generator L. For any differential real value functional Φ, define the Γ operator by

$$\Gamma_\Phi(f) = \frac{1}{2}(L\Phi(f) - \mathrm{d}\Phi(f).Lf), \tag{2.1}$$

where the test function is smooth in $\mathbb{R}^d$ and $\mathrm{d}\Phi(f)$ is understood as

$$\mathrm{d}\Phi(f).g = \lim_{s\to 0} \frac{\Phi(f+sg) - \Phi(f)}{s}. \tag{2.2}$$

The generalized Γ function recovers the classical carré du champ operator in the following way:

- If $\Phi(f) = |f|^2$, then $\Gamma_\Phi(f) = \Gamma(f) = \frac{1}{2}(L(f^2) - 2fLf)$ is the classical Γ operator.

  By direct calculation, for $\Phi(f) = |f|^2$ we have

  $$\mathrm{d}\Phi(f).g = \lim_{s\to 0} \frac{|f+sg|^2 - |f|^2}{s} = 2fg. \tag{2.3}$$

  Introduce the bilinear form corresponding to the Γ operator by

  $$\Gamma(f,g) = \frac{1}{2}(L(fg) - fLg - gLf), \tag{2.4}$$

  then the Γ operator is simply written as $\Gamma(f) = \Gamma(f,f)$.

- If $\Phi(f) = \Gamma(f)$, then $\Gamma_\Phi(f) = \Gamma_2(f) = \frac{1}{2}(L\Gamma(f) - 2\Gamma(f, Lf))$ is the classical Γ_2 operator.

  By direct calculation, for $\Phi(f) = \Gamma(f)$ we have

$$\begin{aligned} \mathrm{d}\Phi(f).g &= \lim_{s\to 0} \frac{\Gamma(f+sg) - \Gamma(f)}{s} \\ &= \lim_{s\to 0} \left(L(fg) + \frac{s}{2}L(g^2) - (fLg + gLf) - sgLg \right) \\ &= 2\Gamma(f, g). \end{aligned}$$

  Hence $\Gamma_\Phi(f) = \Gamma_2(f) = \frac{1}{2}(L\Gamma(f) - 2\Gamma(f, Lf))$. It is also convenient to extend Γ_2 to

$$\Gamma_2(f, g) = \frac{1}{2}(L\Gamma(f, g) - \Gamma(f, Lg) - \Gamma(g, Lf)). \tag{2.5}$$

- If $\Phi(f) = a(f)$ for some scalar function a, then $\Gamma_\Phi(f) = \frac{1}{2}\big(L(a(f)) - a'(f)Lf\big)$.

  By direct calculation,

$$\mathrm{d}\Phi(f).g = a'(f)g.$$

  Hence $\Gamma_\Phi(f) = \frac{1}{2}\big(L(a(f)) - a'(f)Lf\big)$. If L contains the derivation operator $b(x) \cdot \nabla_x$, then it vanishes in the expression of $\Gamma_\Phi(f)$. If L contains the diffusion operator Δ and the scalar function $a(r) = r \log r$, then it is easy to check $\Gamma_\Phi(f) = |\nabla f|^2/f$.

The Γ function is important because it represents the rate of change for the observable functions. Suppose the stochastic process has the semigroup $(P_t)_{t\geqslant 0}$, and define the observable function by

$$\psi(s) = P_s\Phi(P_{t-s}f)(x), \tag{2.6}$$

then $\psi'(s) = 2P_s\Gamma_\Phi P_{t-s}f(x)$. Therefore, for a given function Φ, the curvature condition naturally reveals the long-time behavior of the semigroup $(P_t)_{t\geqslant 0}$.

Lemma 2.1 *Given the differentiable real value functional Φ. The curvature condition $\Gamma_\Phi \geqslant \rho\Phi$ for some $\rho \in \mathbb{R}$, is equivalent to the*

$$\Phi(P_t f) \leqslant e^{-2\rho t}\Phi_t(f), \qquad \forall f \geqslant 0. \tag{2.7}$$

Note the operator P_t is positivity preserving. Also, the Γ operator is linear in the function Φ, and Φ should be nonlinear for Γ to make sense.

We say a scalar function $a \in C^4$ is admissible if a is strictly convex and $1/a''$ is concave. For an admissible function, introduce the a-entropy by

$$\mathrm{Ent}^a_\nu(f) = \nu(a(f)) - a(\nu f) \geqslant 0. \tag{2.8}$$

The a-entropy is equivalent when a is added by a constant. The most common choice of a is $a(r) = r\log r$ with $a''(r) = 1/r$, and the resulting a-entropy is the classical relative entropy. The following lemma estimates Γ_Φ for a specific form of Φ.

Lemma 2.2 *If L is a diffusion operator, C is a linear operator, $a : \mathbb{R}^+ \to \mathbb{R}$ is an admissible function such that $\Phi(f) = a''(f)|Cf|^2$, then*

$$\Gamma_\Phi(f) \geqslant a''(f)Cf.[L, C]f. \tag{2.9}$$

The proof only used the fact that $\alpha = 1/a''$ is concave.

The following lemma makes use of the curvature condition to prove the exponential convergence.

Lemma 2.3 *Suppose the semigroup* $(P_t)_{t\geqslant 0}$ *is ergodic with an invariant distribution* μ*,*

$$0 \leqslant \int \Phi_1(f)\mathrm{d}\mu - \Phi_1\Big(\int f\mathrm{d}\mu\Big) \leqslant c\int \Phi_2(f)\mathrm{d}\mu, \qquad \forall f \geqslant 0, \tag{2.10}$$

and $\Gamma_{\Phi_2} \geqslant \rho\Phi_2 - \beta\Gamma_{\Phi_1}$ *for some* $\beta > 0$*, then the function*

$$W(t) = \beta\Big(\int \Phi_1(P_tf)\mathrm{d}\mu - \Phi_1\Big(\int f\mathrm{d}\mu\Big)\Big) + \int \Phi_2(P_tf)\mathrm{d}\mu \tag{2.11}$$

satisfies $W(t) \leqslant e^{-\frac{2\rho t}{1+\beta c}}W(0)$*. In particular,*

$$\int \Phi_2(P_tf)\mathrm{d}\mu \leqslant (1+\beta c)e^{-\frac{2\rho t}{1+\beta c}}\int \Phi_2(f)\mathrm{d}\mu. \tag{2.12}$$

2.2 Underdamped Langevin dynamics

For the underdamped Langevin dynamics, the infinitesimal generator L (not the one used in the hypocoercivity) is given by

$$Lf(x,y) = \Big[y.\nabla_x - \Big(y + \nabla_x U\Big).\nabla_y + \Delta_y\Big]f(x,y). \tag{2.13}$$

The invariant distribution of the dynamics is

$$\mu(\mathrm{d}x\mathrm{d}y) = \frac{1}{Z}\exp\Big(-U(x) - \frac{|y|^2}{2}\Big). \tag{2.14}$$

Choose the functions $\Phi_1(f) = f\log f$ and

$$\Phi_2(f) = \frac{|(\nabla_x - \nabla_y)f|^2 + |\nabla_y f|^2}{f}, \tag{2.15}$$

then $\Gamma_{\Phi_1}(f) = |\nabla_y f|^2/f$. For the choices of Φ_1, Φ_2, we have

- $0 \leqslant \int \Phi_1(f)\mathrm{d}\mu - \Phi_1\Big(\int f\mathrm{d}\mu\Big) \leqslant c\int \Phi_2(f)\mathrm{d}\mu.$

  In fact, this is equivalent to the log-Sobolev inequality

  $$\mathrm{Ent}_\mu(f) \leqslant c\int \frac{|\nabla f|^2}{f}\mathrm{d}\mu, \tag{2.16}$$

  which holds true due to the classical Bakry–Emery theory.

- $\Gamma_{\Phi_2} \geqslant \rho\Phi_2 - \beta\Gamma_{\Phi_1}.$

In Lemma 2.2, choose $\Phi_{2a} = |(\nabla_x - \nabla_y)f|^2/f$ and $\Phi_{2b} = |\nabla_y f|^2$, we have

$$\Gamma_{\Phi_{2a}} \geqslant \frac{(\nabla_y - \nabla_x)f \cdot (\nabla_y - \nabla_x - \nabla_x^2 U \nabla_y)f}{f}, \tag{2.17}$$

$$\Gamma_{\Phi_{2b}} \geqslant \frac{\nabla_y f \cdot (\nabla_y - \nabla_x)f}{f}. \tag{2.18}$$

Hence

$$\Gamma_{\Phi_2} = \Gamma_{\Phi_{2a}} + \Gamma_{\Phi_{2b}} \geqslant \frac{p|(\nabla_y - \nabla_x)f|^2 - q|\nabla_y f|^2}{f} = p\Phi_2(f) - (p+q)\Gamma_{\Phi_1}. \tag{2.19}$$

The two conditions above imply that the result of Lemma 2.3 holds true. That is, there is exponential convergence in the sense of Φ_2. More precisely,

$$\mathrm{Ent}_\mu(P_t f) \leqslant e^{-\lambda t}\left(\frac{3}{\beta}\int \frac{|\nabla f|^2}{f}\mathrm{d}\mu + \mathrm{Ent}_\mu(f)\right). \tag{2.20}$$

The generalized curvature condition $\Gamma_{\Phi_2} \geqslant \rho\Phi_2 - \beta\Gamma_{\Phi_1}$ plays an important role in the proof.

References

[1] Cédric Villani. Hypocoercivity. *arXiv preprint math/0609050*, 2006.

[2] Pierre Monmarché. Generalized γ calculus and application to interacting particles on a graph. *Potential Analysis*, 50(3):439–466, 2019.