Elliptic Equations

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1 Classical PDE theory

Let $\Omega \subset \mathbb{R}^n$ be an open region and $u \in C^2(\Omega)$. If u satisifies

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0 \tag{1.1}$$

then u is called a harmonic function. A harmonic function is smooth and its derivatives can be bounded by the function itself.

Theorem 1.1 (harmonic function deriative estimation) *Let $u \in C^2(\Omega)$ be a harmonic function on $\Omega \subset \mathbb{R}^n$, then for any ball $B(x,r) \subset \Omega$ and multi-index α with $|\alpha| = k$, we have*

$$|D^\alpha u(x)| \leqslant \frac{C_k}{r^{n+k}} \int_{B(x,r)} |u(y)| \mathrm{d}y \tag{1.2}$$

where

$$C_0 = \frac{1}{\alpha(n)}, \qquad C_k = \frac{(n+k)^{n+k}(n+1)^k}{\alpha(n)(n+1)^{n+1}}$$

The theorem implies the high order derivatives of u can be bounded by the function value itself. We consider the Laplace equation with Dirichlet boundary conditions:

$$\begin{cases} -\Delta u = 0, & x \in B(0,R) \\ \quad u = g, & x \in \partial B(0,R) \end{cases} \tag{1.3}$$

where we assume $g \in C(\partial B(0,R))$. Using the Poisson kernel defined as

$$K(x,y) = \frac{R^2 - |x|^2}{n\alpha(n)R|x-y|^n}, \qquad x \in B(0,R), \quad y \in \partial B(0,R) \tag{1.4}$$

The solution can be explicitly expressed in the following theorem:

Theorem 1.2 (solution of Laplace equation) *Let the ball $B(0,R) \subset \mathbb{R}^n$, and g be a continuous function on $\partial B(0,R)$. The solution of the Laplace equation can be expressed as*

$$u(x) = \int_{\partial B(0,R)} K(x,y) g(y) \mathrm{d}S(y) \tag{1.5}$$

then

1. *$u(x)$ is infinitely differentiable in $B(0,R)$;*
2. *$\Delta u(x) = 0$, $x \in B(0,R)$;*
3. *for any $x_0 \in \partial B(0,R)$, as $x \in B(0,R)$ and $x \to x_0$, $u(x) \to g(x_0)$.*

The theorem implies the solution $u(x) \in C^\infty(\Omega) \cap C(\bar{\Omega})$.

Remark If the weak solution $v(x) \in H^1(\Omega)$ of the Laplace equation is continuous in $\bar{\Omega}$, we can prove that $u(x)$ and $v(x)$ are exactly the same.

2 Functional analysis

Theorem 2.1 (Hölder inequality) *Let $\Omega \subset \mathbb{R}^n$ be an open bounded region. $p, q > 1$ be constants satisfying $1 = \frac{1}{p} + \frac{1}{q}$. For $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, then $uv \in L^1(\Omega)$ and*

$$\left| \int_\Omega uv \mathrm{d}x \right| \leqslant \left(\int_\Omega |u|^p \mathrm{d}x \right)^{\frac{1}{p}} \left(\int_\Omega |v|^q \mathrm{d}x \right)^{\frac{1}{q}} \tag{2.1}$$

When $1 = 1/p + 1/q + 1/r$, the result can be generalized to

$$\left| \int_\Omega uvw \mathrm{d}x \right| \leqslant \left(\int_\Omega |u|^p \mathrm{d}x \right)^{\frac{1}{p}} \left(\int_\Omega |v|^q \mathrm{d}x \right)^{\frac{1}{q}} \left(\int_\Omega |w|^r \mathrm{d}x \right)^{\frac{1}{r}} \tag{2.2}$$

Theorem 2.2 (Riesz representation) *Let H be a Hilbert space with inner product $(\cdot, \cdot)$. Let $f : H \mapsto \mathbb{R}$ be a bounded linear functional, then there exists a unique $u \in H$ such that*

$$\langle f, v \rangle = (u, v), \quad \forall v \in H$$

That is to say, any bounded linear functional in a Hilbert space can be represented by a single element.

Theorem 2.3 (Banach–Alaoglu) *Let X be a normed vector space. Any closed unit ball in X^* is compact in the weak*-topology.*

In particular, the closed unit ball in the Hilbert space is compact in the weak topology, which immediately implies the following theorem.

Theorem 2.4 (Bolzano–Weierstrass) *Let H be a Hilbert space, and $\{x_k\}_{k\geqslant 1}$ be an bounded sequence, then there exists a subsequence $\{x_{k_j}\}_{j\geqslant 1}$ that converges weakly in H.*

Theorem 2.5 (Lax–Milgram) *Let H be a Hilbert space with $a(u, v)$ being a bilinear functional on $H \times H$ satisfying*

- $|a(u, v)| \leqslant M\|u\|\|v\|$, $\forall u, v \in H$;
- $\exists \delta > 0$, $a(u, u) \geqslant \delta\|u\|^2$.

Let f be a bounded linear functional in H, then there exists a unique $u \in H$ such that

$$a(u, v) = \langle f, v \rangle, \quad \forall v \in H$$

Proof Fixing $u \in H$, $a(u, \cdot) : H \mapsto \mathbb{R}$ is a linear functional in H. Then by Reisz representation theorem, there exists $w \in H$ such that

$$a(u, v) = (w, v), \quad \forall w \in H$$

Define the mapping $A : H \mapsto H$ such that $w = Au$, then

$$a(u, v) = (Au, v), \quad \forall u, v \in H$$

then $A : H \mapsto H$ is a linear operator. Now we prove that A is bounded linear operator and a **bijection** in H. Using

$$|(Au, v)| = |a(u, v)| \leqslant M\|u\|\|v\|, \quad \forall u, v \in H$$

By choosing $v = Au$, we have $\|Au\|^2 \leqslant M\|u\|\|Au\|$, thus

$$\|Au\| \leqslant M\|u\|, \quad \forall u \in H$$

which implies A is **bounded**. Also, from

$$\delta\|u\|^2 \leqslant a(u, u) = (Au, u) \leqslant \|Au\|\|u\|$$

we have $\|Au\| \geqslant \delta\|u\|^2$. To prove A is bijective, we need to verify

- $R(A)$ is **closed**.

  Consider a convergent sequence $Au_n \in R(A)$, now we prove that $Au_n \to Au$ for some $u \in H$. To prove this result, notice that

  $$\lim_{m,n\to\infty} \|Au_m - Au_n\| = 0$$

  and $\|A(u_m - u_n)\| \geqslant \delta\|u_m - u_n\|$, we have

  $$\lim_{m,n\to\infty} \|u_m - u_n\| = 0$$

  which implies $\{u_n\}$ is convergent in H. Let $u_n \to u$, then $Au_n \to Au \in R(A)$.

- $R(A)^\perp = \emptyset$.

  Otherwise, let $w \in R(A)^\perp$ and $w \neq 0$, then

  $$(w, Au) = 0, \quad \forall u \in H$$

  which implies $a(w,u) = 0$, $\forall u \in H$. Choosing $u = w$, we have $a(w,w) = 0 \Rightarrow w = 0$, contradiction.

The results above imply $R(A) = H$, thus A is bijective. The equation $Au = f$ can be solved explicitly as $u = A^{-1}f$.

Remark The key of the Lax-Milgram theorem is the coercivity of the functional. The theorem does not require the symmetry of the functional $a(\cdot,\cdot)$.

3 Sobolev spaces

3.1 Basic properties

Let $\Omega \subset \mathbb{R}^n$ be an open bounded region. For the multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$, $\partial^\alpha u$ denotes the α-weak derivative of u.

Definition 3.1 (Sobolev space) *Let $m \geqslant 0$ be an integer and $p > 0$. The function space*

$$W = \{u \in L^p(\Omega) : \partial^\alpha u \in L^p(\Omega), |\alpha| \leqslant m\}$$

is the Sobolev space $W^{m,p}(\Omega)$. The Sobolev space is a Banach space with norm

$$\|u\|_{m,p} = \left(\sum_{0\leqslant|\alpha|\leqslant m} \int_\Omega |\partial^\alpha u|^p \mathrm{d}x\right)^{\frac{1}{p}}$$

and the correspondong seminorm is

$$|u|_{m,p} = \left(\sum_{|\alpha|=m} \int_\Omega |\partial^\alpha u|^p \mathrm{d}x\right)^{\frac{1}{p}}$$

which only involves derivatices of order m. $W_0^{m,p}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$.

Theorem 3.1 (Poincare-Friedrichs) *Let $\Omega \subset \mathbb{R}^n$ be an open bounded region. There exists a constant $K(n,m,p)$ that*

$$|u|_{m,p} \leqslant \|u\|_{m,p} \leqslant K(n,m,p)|u|_{m,p}, \quad \forall u \in W_0^{m,p}$$

The theorem implies the norm and the seminorm in $W_0^{m,p}$ are equivalent.

3.2 Embedding theorem

Let X, Y be two Banach spaces. We say X is continuously embedded in Y, or denoted by $X \hookrightarrow Y$, if $X \subset Y$ and there exists a constant C such that

$$\|x\|_Y \leqslant C\|x\|_X, \quad \forall x \in X$$

Recall that the norm in the Sobolev space $W^{m,p}(\Omega)$ is

$$\|u\|_{m,p} = \left(\sum_{|\alpha| \leqslant m} \int_\Omega |\partial^\alpha u|^p \mathrm{d}x \right)^{\frac{1}{p}}$$

and the norm in $C(\bar{\Omega})$ is

$$\|u\|_{C(\bar{\Omega})} = \max_{\bar{\Omega}} |u|$$

Theorem 3.2 (Sobolev embedding) *Let $\Omega \subset \mathbb{R}^n$ be an open bounded region in $\mathbb{R}^n$. If $\partial\Omega$ is Lipschitz continuous, then*

- *If $m < n/p$, $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $1 \leqslant 1 \leqslant \frac{np}{n-mp}$;*
- *If $m = n/p$, $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $1 \leqslant q < +\infty$;*
- *If $m > n/p$, $W^{m,p}(\Omega) \hookrightarrow C(\bar{\Omega})$.*

In particular, when $m = 1$ and $p = 2$, we have

$$H^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$$

where $2^* = \frac{2n}{n-2}$.

3.3 H^{-1} space

Now we state the definition of H^{-1} space (see Evans Section 5.9).

Definition 3.2 ($\boldsymbol{H^{-1}}$ space) *Let $\Omega \subset \mathbb{R}^n$ be an open bounded region. $H^{-1}(\Omega)$ is defined as the dual space of $H_0^1(\Omega)$.*

It's easy to see

$$H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$$

The functions in $H^{-1}(\Omega)$ can be characterized as follows:

Theorem 3.3 (characterization of H^{-1} space) *Assume $u \in H^{-1}(\Omega)$, then there exists functions $f^0, f^1, \cdots, f^n \in L^2(\Omega)$ such that*

$$\langle f, v\rangle = \int_\Omega \left(f^0 v + \sum_{i=1}^{n} f^n v_{x_i} \right) \mathrm{d}x \qquad (*)$$

Furthermore,

$$\|f\|_{H^{-1}(\Omega)} = \left\{ \int_\Omega \sum_{i=0}^{n} |f^i|^2 \mathrm{d}x : f^0, f^1, \cdots, f^n \text{ satisfying } (*) \right\}$$

Therefore, we can write a function $f \in H^{-1}(\Omega)$ in the form

$$f = f^0 - \sum_{i=1}^{n} f^i_{x_i}$$

for $f^0, f^1, \cdots, f^n \in L^2(\Omega)$.

4 Second-order elliptic equations

4.1 Weak solution of the elliptic equation

Let $\Omega \subset \mathbb{R}^n$ be an open bounded region with C^1 boundary. Consider the second-order elliptic equation in Ω with Dirichlet boundary conditions

$$\begin{cases} -D_j(a^{ij} D_i u) = f - D_i f^i, & x \in \Omega \\ u = 0, & x \in \partial\Omega \end{cases} \tag{4.1}$$

We ultimate the following assumptions (A):

1. $a_{ij} \in L^\infty(\Omega)$ and there exists constants $\lambda, \Lambda > 0$ that

$$\lambda |\xi|^2 \leqslant \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \leqslant \Lambda |\xi|^2, \quad x \in \Omega, \quad \xi \in \mathbb{R}^n; \tag{4.2}$$

2. $c(x) \geqslant 0$ and $c \in L^{\frac{n}{2}}(\Omega)$;
3. $f, f^i \in L^2(\Omega)$, $i = 1, \cdots, n$.

Remark The right hand side of (4.1) is in the H^{-1} space. Written in the form of $f - D_i f^i$ makes it easier to write the weak formulation.

Now we define the weak solution of the elliptic equation (4.1).

Definition 4.1 (weak solution) *The weak solution of the second-order elliptic equation (4.1) is*

$$\int_\Omega \left(a^{ij} D_i u D_i v + cuv \right) \mathrm{d}x = \int_\Omega \left(fv + f^i D_i v \right) \mathrm{d}x, \qquad \forall v \in C_0^\infty(\Omega) \tag{4.3}$$

It can be verified that the strong solution of (4.1) is also the weak solution. Under the assumptions above, we can prove that the elliptic equation (4.1) has a unique weak solution in $H_0^1(\Omega)$. Note that for Dirichlet boundary conditions (homogeneous or not), the test function space is $H_0^1(\Omega)$, i.e., the trace of the function must be zero.

Theorem 4.1 (existence of weak solution) *Under the assumptions (A), the elliptic equation (4.1) has a unique weak solution in H_0^1. That is, there exists a unique $u \in H_0^1(\Omega)$ satisfying (4.3).*

Proof The proof of the existence is based on the Lax-Milgram theorem. Define the bilinear functional $a(\cdot,\cdot)$ in $H_0^1(\Omega)$:

$$a(u,v) := \int_\Omega \left(a^{ij} D_i u D_j v + cuv\right) \mathrm{d}x, \qquad u, v \in H_0^1(\Omega) \tag{4.4}$$

First we prove that $a(\cdot,\cdot)$ is **bounded**. From

$$\left|\int_\Omega a^{ij} D_i u D_j v \mathrm{d}x\right| \leqslant \int_\Omega |a^{ij}||D_i u||D_i v| \mathrm{d}x \leqslant \Lambda \left(\int_\Omega |Du|^2 \int_\Omega |Dv|^2\right)^{\frac{1}{2}} = \Lambda \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}$$

we have

$$\left|\int_\Omega a^{ij} D_i u D_j v \mathrm{d}x\right| \leqslant \Lambda \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \tag{4.5}$$

From the Hölder inequality, we have

$$\left|\int_\Omega cuv \mathrm{d}x\right| \leqslant \|c\|_{L^{\frac{n}{2}}(\Omega)} \|u\|_{L^{2^*}(\Omega)} \|v\|_{L^{2^*}(\Omega)} \tag{4.6}$$

where $2^* = \frac{2n}{n-2}$. here we used

$$\frac{2}{n} + \frac{1}{2^*} + \frac{1}{2^*} = \frac{2}{n} + \frac{n-2}{2n} \cdot 2 = 1$$

Using the Sobolev embedding result $H^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$, we obtain

$$\left|\int_\Omega cuv \mathrm{d}x\right| \leqslant C \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \tag{4.7}$$

Using (4.5)(4.7) we conclude $a(\cdot,\cdot)$ is a bounded bilinear functional.

Next we prove that $a(\cdot,\cdot)$ is **coercive**. It can be verified that

$$a(u,u) = \int_\Omega \left(a^{ij} D_i u D_j u + cu^2\right) \mathrm{d}x \geqslant \lambda \int_\Omega |Du|^2 \mathrm{d}x \tag{4.8}$$

Using the Poincáre inequality, we have

$$\int_\Omega |Du|^2 \mathrm{d}x \geqslant c \int_\Omega |u|^2 \mathrm{d}x \tag{4.9}$$

and we obtain

$$a(u,u) \geqslant c\|u\|^2_{H^1(\Omega)} \tag{4.10}$$

thus $a(\cdot,\cdot)$ is coercive.

Now define the linear functional $F(\cdot)$ in $H_0^1(\Omega)$ as

$$F(v) = \int_\Omega \Big(fv + f^i D_i v\Big)\mathrm{d}x \tag{4.11}$$

then

$$|F(v)| \leqslant \|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} + \left\|f^i\right\|_{L^2(\Omega)}\|v\|_{H^1(\Omega)} \leqslant C\|v\|_{H^1(\Omega)} \tag{4.12}$$

thus F is a **bounded** linear function.

Finally, by the Lax-Milgram theorem, the equation $a(u,v) = F(v)$ has the unique solution in $H_0^1(\Omega)$. That is, the weak formulation (4.3) has a unique solution in $H_0^1(\Omega)$.

4.2 Weak maximum principle

First we introduce the lemma of De Giorgi iterations.

Lemma 4.1 (De Giorgi) *Let $\varphi(t)$ be a nonnegative and decreasing function satisfying*

$$\varphi(h) \leqslant \frac{C}{(h-k)^\alpha}[\varphi(k)]^\beta, \qquad h > k \geqslant k_0, \tag{4.13}$$

where $\alpha > 0, \beta > 1$, then $\varphi(k_0 + d) = 0$, where

$$d = C^{\frac{1}{\alpha}}[\varphi(k_0)]^{\frac{\beta-1}{\alpha}} 2^{\frac{\beta}{\beta-1}}. \tag{4.14}$$

The lemma implies any function $\varphi(t)$ satsifying the condition must vanish for sufficiently large t.

Proof The proof is based on the following sequence $\{k_s\}_{s\geqslant 0}$:

$$k_s = k_0 + d - \frac{d}{2^s}, \qquad s = 0, 1, \cdots$$

Now we define the weak upper (lower) solutions as follows.

Definition 4.2 (weak upper (lower) solution) $u \in H^1(\Omega)$ *is called the weak upper (lower) solution of the elliptic equation (4.1), if for all $\varphi \in C_0^\infty(\Omega)$ and $\varphi \geqslant 0$, we have*

$$\int_\Omega \Big(a^{ij} D_i u D_j \varphi + cu\varphi\Big)\mathrm{d}x \geqslant (\leqslant) \int_\Omega \Big(f\varphi + f^i D_i\varphi\Big)\mathrm{d}x \tag{4.15}$$

Remark The difference between the weak upper solution (4.15) and the weak solution (4.3) is that the test function in (4.15) has to nonnegative. If $u \in H_0^1(\Omega)$ and u is both a weak upper and lower solution, u is the weak solution. The corresponding strong form of the weak lower solution is

$$-D_j(a^{ij} D_i u) + cu \leqslant f - D_i f^i \tag{4.16}$$

which is exactly the definition of the subharmonic function.

Theorem 4.2 (weak maximum principle) *If the assumption (A) holds, and if $u \in H^1(\Omega)$ is a weak lower solution of (4.1), then for any $p > n$,*

$$\operatorname*{ess\,sup}_{\Omega} u \leqslant \sup_{\partial\Omega} u^+ + C\Big(\|f\|_{L^{p_*}} + \big\|f^i\big\|_{L^p}\Big)|\Omega|^{\frac{1}{n}-\frac{1}{p}}, \tag{4.17}$$

where

$$p_* = \frac{np}{n+p} < p.$$

Proof Take $k_0 = \sup_{\partial\Omega} u^+ \geqslant 0$, then for $k > k_0$, $\varphi = (u-k)^+ \in H_0^1(\Omega)$. Since u is a weak lower solution, by choosing $\varphi = (u-k)^+$ in (4.15), we have

$$\int_\Omega \Big(a^{ij} D_i u D_j (u-k)^+ + cu(u-k)^+\Big)\mathrm{d}x \leqslant \int_\Omega \Big(f(u-k)^+ + f^i D_i(u-k)^+\Big)\mathrm{d}x, \quad \forall k > k_0 \tag{4.18}$$

Note that in (4.18), $D_i(u-k) = D_i(u-k)^+ - D_i(u-k)^-$ and thus

$$D_i(u-k)D_j(u-k)^+ = D_i(u-k)^+ D_j(u-k)^+$$

and we have

$$\begin{aligned}\int_\Omega a^{ij} D_i u D_j (u-k)^+ \mathrm{d}x &= \int_\Omega a^{ij} D_i(u-k) D_j(u-k)^+ \mathrm{d}x \\ &= \int_\Omega a^{ij} D_i(u-k)^+ D_j(u-k)^+ \mathrm{d}x \\ &\geqslant \lambda \sum_{i=1}^n \int_\Omega \big(D_i(u-k)^+\big)^2 \mathrm{d}x = \lambda \big\|D(u-k)^+\big\|^2_{L^2(\Omega)}\end{aligned}$$

that is,

$$\int_\Omega a^{ij} D_i u D_j (u-k)^+ \mathrm{d}x \geqslant \lambda \|D(u-k)_+\|^2_{L^2(\Omega)} \tag{4.19}$$

Also, note that in (4.18), $u(u-k)^+ \geqslant 0$ for each $x \in \Omega$. If $u(u-k)^+ < 0$ for some $x \in \Omega$, using $(u-k)^+ \geqslant 0$ we have $u(x) < 0$ and $(u(x)-k)^+ > 0$. This implies $u(x) > k \geqslant 0$, contradiction! Therefore,

$$\int_\Omega cu(u-k)^+ \mathrm{d}x \geqslant 0 \tag{4.20}$$

From (4.18)(4.19)(4.20) we conclude

$$\lambda \big\|D(u-k)^+\big\|^2_{L^2(\Omega)} \leqslant \int_\Omega \Big(f(u-k)^+ + f^i D_i(u-k)^+\Big)\mathrm{d}x, \quad \forall k > k_0 \tag{4.21}$$

Now we estimate the RHS of (4.21). Using the Hölder inequality,

$$\left|\int_\Omega f(u-k)^+ \mathrm{d}x\right| \leqslant \|f\|_{L^{p_*}(\Omega)} \big\|(u-k)^+\big\|_{L^{2^*}(\Omega)} |A_k|^{\frac{1}{2}-\frac{1}{p}} \tag{4.22}$$

where $A_k := \{x \in \Omega : u(x) > k\}$ and we used

$$\frac{1}{p_*} + \frac{1}{2^*} = \frac{n+p}{np} + \frac{n-2}{2n} + \frac{1}{2} - \frac{1}{p} = 1$$

From the embedding $H^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ and the Poincáre inequality, for $(u-k)^+ \in H_0^1(\Omega)$ we have

$$\left\|(u-k)^+\right\|_{L^{2^*}(\Omega)} \leqslant C\left\|(u-k)^+\right\|_{H^1(\Omega)} \leqslant C\left\|D(u-k)^+\right\|_{L^2(\Omega)} \tag{4.23}$$

hence (4.21) gives

$$\left|\int_\Omega f(u-k)^+ \mathrm{d}x\right| \leqslant C\|f\|_{L^{p_*}(\Omega)}\left\|D(u-k)^+\right\|_{L^2(\Omega)}|A_k|^{\frac{1}{2}-\frac{1}{p}} \tag{4.24}$$

Similarly,

$$\left|\int_\Omega f^i D_i(u-k)^+ \mathrm{d}x\right| \leqslant \sum_{i=1}^n \left\|f^i\right\|_{L^p(\Omega)}\left\|D(u-k)^+\right\|_{L^2(\Omega)}|A_k|^{\frac{1}{2}-\frac{1}{p}} \tag{4.25}$$

(4.24)(4.25) yield

$$\left|\int_\Omega \Big(f(u-k)^+ + f^i D_i(u-k)^+\Big)\mathrm{d}x\right| \leqslant C\Big(\|f\|_{L^{p_*}} + \sum_{i=1}^n \left\|f^i\right\|_{L^p}\Big)\left\|D(u-k)^+\right\|_{L^2}|A_k|^{\frac{1}{2}-\frac{1}{p}} \tag{4.26}$$

Let $F = \|f\|_{L^{p_*}} + \sum_{i=1}^n \left\|f^i\right\|_{L^p}$ be a constant. (4.21)(4.26) give

$$\left\|D(u-k)^+\right\|_{L^2(\Omega)} \leqslant CF|A_k|^{\frac{1}{2}-\frac{1}{p}} \tag{4.27}$$

Again from the Sobolev embedding and the Poincáre inequality,

$$\left\|(u-k)^+\right\|_{L^{2^*}(\Omega)} \leqslant CF|A_k|^{\frac{1}{2}-\frac{1}{p}} \tag{4.28}$$

For any $h > k > k_0$, we have

$$(h-k)^{2^*}|A_k| \leqslant \Big(CF|A_k|^{\frac{1}{2}-\frac{1}{p}}\Big)^{2^*} \tag{4.29}$$

Now consider decreasing function A_k on $[k_0, +\infty)$. Applying the De Giorgi iteration lemma, $A_{k_0+d} = 0$, where

$$d = CF|A_{k_0}|^{\frac{1}{n}-\frac{1}{p}} 2^{\frac{n(p-2)}{2(p-n)}} \leqslant CF|\Omega|^{\frac{1}{n}-\frac{1}{p}} \tag{4.30}$$

which implies

$$\operatorname*{ess\,sup}_\Omega u \leqslant k_0 + d \leqslant \sup_{\partial\Omega} u^+ + C\Big(\|f\|_{L^{p_*}} + \left\|f^i\right\|_{L^p}\Big)|\Omega|^{\frac{1}{n}-\frac{1}{p}} \tag{4.31}$$

and concludes our result.

Remark The maximum principle requires higher integrity on f than the existence of the weak solution. The maximum principle in the weak solution is the same with the harmonic function.

In the case of $n = 2$, the proof is a little different.

$$\left|\int_\Omega f(u-k)^+ \mathrm{d}x\right| \leqslant \left\|D(u-k)^+\right\|_{L^2}\|f\|_{L^{p_*}}|A_k|^{\frac{1}{2}-\frac{1}{p}}$$

4.3 Variantional principle

We can derive the existence of the weak solution from the variational principle. That is to say, minimize some functional in a proper function space. For simplicity, let $\Omega \subset \mathbb{R}^n$ be an open bounded region and consider the elliptic equation with Dirichlet boundary conditions

$$\begin{cases} -D_j(a^{ij}D_iu) = 0, & x \in \Omega \\ u = g, & x \in \partial\Omega \end{cases} \tag{4.32}$$

where the coefficients a^{ij} satisfy the uniform elliptic condition:

$$\lambda|\xi|^2 \leqslant \sum_{i,j=1}^{n} a^{ij}\xi_i\xi_j \leqslant \Lambda|\xi|^2 \tag{4.33}$$

Now we aim to minimize the functional

$$F(u) = \frac{1}{2}\int_\Omega a^{ij}D_iuD_ju\mathrm{d}x, \qquad u \in H^1(\Omega) \tag{4.34}$$

within the function space

$$W = \{u \in H^1(\Omega) : u - g \in H_0^1(\Omega)\} \tag{4.35}$$

The proof is consists of 3 parts:

1. The functional F has minimizer in W;
2. The minimizer yields the weak solution of (4.32).
3. The minimizer of F is unique.

First we prove the existence of the minimizer. Suppose $\{u_n\} \subset W$ satisfies

$$\lim_{n\to\infty} F(u_n) = \inf_{u\in W} F(u)$$

From

$$F(u) \geqslant \frac{\lambda}{2}\int_\Omega |Du|^2\mathrm{d}x$$

we know that $\|Du_n\|_{L^2}$ is bounded, thus $\|D(u_n - g)\|_{L^2}$ is bounded. From the Poincáre inequality, $\|u_n - g\|_{H^1}$ is bounded. Thus there exists a subsequence $\{u_{n_k}\}$ and $u^* \in H^1(\Omega)$ such that

$$u_{n_k} \to u^* \ \text{ in } L^2(\Omega)$$

$$u_{n_k} \rightharpoonup u^* \ \text{ in } H^1(\Omega)$$

thus u^* satisfies

$$F(u^*) \leqslant \varliminf_{n\to\infty} F(u_n) = \inf_{u\in W} F(u)$$

which implies u^* is a minimizer.

Note that the Gauteux derivative of F is

$$\frac{\delta F}{\delta v}(u) = \int_\Omega a^{ij}D_iuD_jv\mathrm{d}x, \qquad v \in H_0^1(\Omega)$$

If u^* is a minimizer, we have

$$\int_\Omega a^{ij} D_i(u^*) D_j v \mathrm{d}x = 0, \quad \forall v \in H_0^1(\Omega)$$

thus u^* is a weak solution.

If u^* and u^{**} are both weak solutions, then

$$\frac{\delta F}{\delta v}(u^*) = \frac{\delta F}{\delta v}(u^{**}) = 0, \quad \forall v \in H_0^1(\Omega)$$

Consider the function $f(t) := F(tu^* + (1-t)u^{**})$, then f is quadratic and $f(t)$ is has 2 stationary points at $t = 0, 1$. Hence f is constant, which implies $D(u^* - u^{**}) = 0$. $u^* = u^{**}$.

$$F = \int_\Omega |\nabla u|^p \mathrm{d}x + \int_\Omega fu, \quad u \in W^{1,p}$$

4.4 Regularity of the solution

The regularity the weak solution of the elliptic equation can be derived given techincal assumptions. Consider the elliptic equation

$$-D_i(a^{ij} D_j u) + b^i D_i u + cu = f \tag{4.36}$$

with appropriate Dirichlet boundary conditions. Under the conditions of (A), we make stronger assumptions (B) on the coefficients:

1. $a^{ij} \in W^{1,\infty}(\Omega)$;
2. $b^i, c \in L^\infty(\Omega)$;
3. $f \in L^2(\Omega)$.

Theorem 4.3 ($\boldsymbol{H^2}$ regularity) *Let $u \in H^1(\Omega)$ be the weak solution. Then for any $\Omega' \subset\subset \Omega$, $u \in H^2(\Omega')$, and*

$$\|u\|_{H^2(\Omega')} \leqslant C\Big(\|u\|_{H^1(\Omega)} + \|f\|_{L^2(\Omega)}\Big) \tag{4.37}$$

where the constant C depends on n, λ, $\left\|a^{ij}\right\|_{W^{1,\infty}}$, $\left\|b^i\right\|_{L^\infty}$, $\|c\|_{L^\infty}$ and $\operatorname{dist}(\Omega', \partial\Omega)$.

When we make no assumptions on the shape of Ω, the constant C might grow large as Ω' enlarges. The proof is based on the difference function. Given the direction $v \in \mathbb{R}^n$ with $|v| = 1$, define the translation operator

$$\tau_h u(x) = u(x + hv), \quad h \in \mathbb{R}$$

and the difference operator in the direction v,

$$\Delta_h u(x) = \frac{1}{h}\big(u(x + hv) - u(x)\big), \quad h \in \mathbb{R}$$

With this notation, we can write

$$\Delta_{-h} u(x) = \frac{1}{h}\big(u(x - hv) - u(x)\big)$$

We can verify the following identity

$$\Delta_h(uv) = \tau_h \Delta_h v + (\Delta_h u)v$$

Proof Set $q = f - b^i D_i u - cu \in L^2(\Omega)$, then the weak formulation can be written as

$$\int_\Omega a^{ij} D_i u D_j \varphi \mathrm{d}x = \int_\Omega q\varphi \mathrm{d}x, \quad \forall \varphi \in H_0^1(\Omega) \tag{4.38}$$

Choose the test function $\varphi = \Delta_{-h} v$

Let $v \in H_0^1(\Omega)$ with compact support in Ω. For sufficiently small h, $\Delta_h v$ is well-defined. By choosing the test function $\varphi = \Delta_{-h} v \in H_0^1(\Omega)$, we have

$$\int_\Omega a^{ij} D_i u D_j \Delta_{-h} v \mathrm{d}x = \int_\Omega q \Delta_{-h} v_{\mathrm{d}} x \tag{4.39}$$

which implies

$$\int_\Omega \Delta_h (a^{ij} D_i u) D_i v \mathrm{d}x = -\int_\Omega q \Delta_{-h} v \mathrm{d}x \tag{4.40}$$

Using

$$\Delta_h (a^{ij} D_i u) = \tau_h a^{ij} \Delta_h D_i u + D_i u \Delta_h a_{ij} \tag{4.41}$$

hence we obtain

$$\int_\Omega \tau_h a^{ij} \Delta_h D_i u D_i v \mathrm{d}x = -\int_\Omega \Big(\Delta_h a^{ij} D_i u D_j v + q \Delta_{-h} v \Big) \mathrm{d}x \tag{4.42}$$

Note that

$$\left| \int_\Omega \Delta_h a^{ij} D_i u D_j v \mathrm{d}x \right| \leqslant C \big\| D a^{ij} \big\|_{L^\infty} \|u\|_{H^1} \|Dv\|_{L^2}$$

and

$$\left| \int_\Omega q \Delta_{-h} v \mathrm{d}x \right| \leqslant C \|q\|_{L^2} \|Dv\|_{L^2}, \qquad \|q\|_{L^2} \leqslant C\|f\|_{L^2} + C\|u\|_{H^1}$$

Hence

$$\int_\Omega \tau_h a^{ij} \Delta_h D_i u D_j v \mathrm{d}x \leqslant C \Big(\|u\|_{H^1} + \|f\|_{L^2} \Big) \|Dv\|_{L^2} \tag{4.43}$$

Further choose $v = \eta^2 \Delta_h u$

For simplicity, let $F = \|u\|_{H^1} + \|f\|_{L^2}$ be the constant. Choosing $\eta \in C_0^\infty$ with $\eta = 1$ for $x \in \Omega'$ and $v = \eta^2 \Delta_h u$, we have

$$Dv = D(\eta^2 \Delta_h u) = \eta(\eta D \Delta_h u + \Delta_h u D\eta)$$

hence

$$\begin{aligned} & \int_\Omega \eta^2 \tau_h a^{ij} D_i \Delta_h u D_j \Delta_h u \mathrm{d}x + 2 \int_\Omega \eta \tau_h a^{ij} D_i \Delta_h u (D_i \eta) \Delta_h u \mathrm{d}x \\ \leqslant \; & C \Big(\|u\|_{H^1} + \|f\|_{L^2} \Big) \Big(\|\eta D \Delta_h u\|_{L^2} + 2\|\Delta_h u D\eta\|_{L^2} \Big) \\ \leqslant \; & CF \|\eta D \Delta_h u\|_{L^2} + CF^2 \end{aligned}$$

That is,

$$\int_\Omega \eta^2 \tau_h a^{ij} D_i \Delta_h u D_j \Delta_h u \mathrm{d}x + 2\int_\Omega \eta \tau_h a^{ij} D_i \Delta_h u (D_i \eta) \Delta_h u \mathrm{d}x \leqslant CF \|\eta D \Delta_h u\|_{L^2} + CF^2 \tag{4.44}$$

Now we estimate the LHS of (4.44).

$$\int_\Omega \eta^2 \tau_h a^{ij} D_i \Delta_h u D_j \Delta_h u \mathrm{d}x \geqslant \lambda \int_\Omega |\eta \Delta_h D u|^2 \mathrm{d}x$$

$$2\left| \int_\Omega \eta \tau_h a^{ij} D_i \Delta_h u (D_i \eta) \Delta_h u \mathrm{d}x \right| \leqslant C \int_\Omega |D\eta|^2 |\Delta_h u|^2 \mathrm{d}x + \frac{\lambda}{2} \int_\Omega |\eta D_i \Delta_h u|^2 \mathrm{d}x$$

Thus

$$\int_\Omega \eta^2 \tau_h a^{ij} D_i \Delta_h u D_j \Delta_h u \mathrm{d}x \geqslant \|\eta D \Delta_h u\|_{L^2}^2 - CF^2$$

Therefore (4.44) gives

$$\|\eta D \Delta_h u\|_{L^2}^2 \leqslant CF \|\eta D \Delta_h u\|_{L^2} + CF^2$$

which implies

$$\|\eta D \Delta_h u\|_{L^2} \leqslant CF \tag{4.45}$$

and its exactly what need.

If the boundary $\partial\Omega$ is smooth, we can derive more accurate estimations.

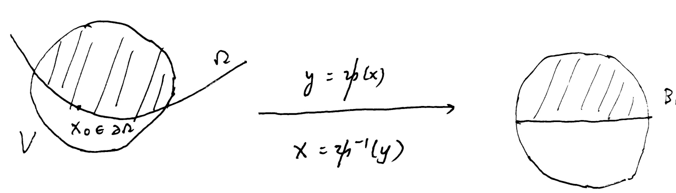
Theorem 4.4 (H^2 regularity) *Additionally suppose $\partial\Omega \in C^2$, $g \in H^2(\Omega)$ and $u - g \in H_0^1(\Omega)$, where $u \in H^1(\Omega)$ is a weak solution of the elliptic equation. Then $u \in H^2(\Omega)$ and*

$$\|u\|_{H^2(\Omega)} \leqslant C\Big(\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} + \|g\|_{H^2(\Omega)} \Big) \tag{4.46}$$

where the constant C depends on $n, \lambda, \left\|a^{ij}\right\|_{W^{1,\infty}}, \left\|b^i\right\|_{L^\infty}, \|c\|_{L^\infty}$ and $\partial\Omega$.

The proof of the global estimation of the H^2-norm is based on the technique that the boundary of $\partial\Omega$ can be stretched to be plane.

Proof First assume $g = 0$. Let $x^0 \in \partial\Omega$ and $\psi : V \mapsto B_1$ maps $V \cap \Omega$ to B_1^+, as shows below.



By choosing the test function $\varphi \in C_0^\infty(\Omega)$, the weak formulation is given by

$$\int_\Omega a^{ij} D_i u D_j \varphi = \int_\Omega q \varphi \mathrm{d}x \tag{4.47}$$

where $q = f - cu \in L^2(\Omega)$. If we choose $\varphi \in C_0^\infty(V \cap \Omega)$, then

$$\int_{V\cap\Omega} \left(a^{ij} D_i u D_j \varphi + cu\varphi\right) \mathrm{d}x = \int_{V\cap\Omega} f\varphi \mathrm{d}x \tag{4.48}$$

Using the transformation $y = \psi(x)$, we reformulate the expression in the coordinates of y.

$$D_i u = \frac{\partial u}{\partial x_i} = \frac{\partial u}{\partial y_k}\frac{\partial y_k}{\partial x_i} = \frac{\partial y_k}{\partial x_i}\tilde{D}_k u$$

Similarly

$$D_j \varphi = \frac{\partial y_l}{\partial x_j}\tilde{D}_l \varphi$$

Hence

$$\int_{V\cap\Omega} a^{ij} D_i u D_j \varphi \mathrm{d}x = \int_{B_1^+} \tilde{a}^{kj} \tilde{D}_k u \tilde{D}_l \varphi \mathrm{d}y \tag{4.49}$$

where we define the new coefficent

$$\tilde{a}^{kl} = J a^{ij} \frac{\partial y_k}{\partial x_i}\frac{\partial y_l}{\partial x_j}$$

Similarly,

$$s \tag{4.50}$$

5 Schauder estimate: [GT]

In this part we discuss of the regularity of the solution for the Poisson equation

$$\Delta u = f \tag{5.1}$$

When $f(x)$ is Hölder continuous, we hope to prove that the solution is also continous. Some notations:

- **(boundness of $D^k u$)** $[u]_{k,0;\Omega} = |D^k u|_{0;\Omega} = \sup_{|\beta|=k} \sup_{\Omega} |D^\beta u|$
- **(continuity of $D^k u$)** $[u]_{k,\alpha;\Omega} = [D^k u]_{\alpha;\Omega} = \sup_{|\beta|=k} [D^\beta u]_{\alpha;\Omega}$

The seminorms indicate the corresponding norms

- **(boundness of $D^k u$)** $\|u\|_{C^k(\Omega)} = |u|_{k;\Omega} = |u|_{k,0;\Omega} = \sum_{j=0}^{k} [u]_{j,0;\Omega} = \sum_{j=0}^{k} |D^j u|_{0;\Omega}$
- **(continuity of $D^k u$)** $\|u\|_{C^{k,\alpha}(\Omega)} = |u|_{k,\alpha;\Omega} = |u|_{k;\Omega} + [u]_{k,\alpha;\Omega} = |u|_{k;\Omega} + [D^k u]_{\alpha;\Omega}$

Compared to the norm in $C^k(\Omega)$, the norm in $C^{k,\alpha}$ includes the α-Hölder continuity of $D^k u$. The scaled norms of u are also useful in applications. (The prime notation denotes the scaling)

- **(scaled boundness of $D^k u$)** $\|u\|'_{C^k(\Omega)} = |u|'_{k;\Omega} = \sum_{j=0}^{k} d^j [u]_{j,0;\Omega} = \sum_{j=0}^{k} d^j |D^j u|_{0;\Omega}$

- **(scaled continuity of $D^k u$)** $\|u\|'_{C^{k,\alpha}} = |u|'_{k,\alpha;\Omega} = |u|'_{k;\Omega} + d^{k+\alpha}[u]_{k,\alpha;\Omega} = |u|'_{k;\Omega} + d^{k+\alpha}[D^k u]_{\alpha;\Omega}$

It is notable that the definition of $\|u\|'_{C^k(\Omega)}$ and $\|u\|'_{C^{k,\alpha}(\Omega)}$ both has scaling variance, when the underlying region is scaled.

5.1 Classical theory of the Newtonian potential

This part is based on the Chapters 2,4 of [GT][1]. Recall that the fundamental solution to the Laplace equation is

$$\Gamma(x) = \begin{cases} \dfrac{1}{n(2-n)\omega_n}|x|^{2-n}, & n > 2 \\ \dfrac{1}{2\pi}\log|x|, & n = 2 \end{cases} \tag{5.2}$$

which satisfies the equation

$$\Delta\Gamma(x) = \delta(x) \tag{5.3}$$

The definition of $\Gamma(x)$ is slightly different from others where $\Gamma(x)$ is set to satisfy $\Delta\Gamma(x) = -\delta(x)$. By direct calculation, the derivatives of $\Gamma(x)$ are given by

Theorem 5.1 (Derivatives of the fundamental solution) *The first and second order derivatives of* $\Gamma(x)$ *are given by*

$$D_i\Gamma(x) = \frac{1}{n\omega_n} x_i |x|^{-n}$$
$$D_{ij}\Gamma(x) = \frac{1}{n\omega_n}\left(|x|^2\delta_{ij} - n x_i x_j\right)|x|^{-n-2}$$

The theorem implies the derivatives are bounded by

$$|D_i\Gamma(x)| \leqslant \frac{1}{n\omega_n}|x|^{1-n}$$
$$|D_{ij}\Gamma(x)| \leqslant \frac{1}{\omega_n}|x|^{-n}$$

This result implies that

$$\int_{B_R} |D\Gamma(x)|\mathrm{d}x < +\infty, \qquad \int_{B_R} |D^2\Gamma(x)|\mathrm{d}x = +\infty$$

That is, $D\Gamma$ is integrable but $D^2\Gamma$ is not. Now we consider the Poisson equation $\Delta u = f$ in a regular region $\Omega \subset \mathbb{R}^n$. Let's first define the **Newtonian potential** corresponding to f by

$$w(x) = \int_\Omega \Gamma(x-y) f(y)\mathrm{d}y, \qquad x \in \mathbb{R}^n \tag{5.4}$$

Some basic notations on $w(x)$:

- Although $f(x)$ is defined in Ω, the Newtonian potential $w(x)$ is defined in $\mathbb{R}^b$.

[1] [GT] denotes the classical textbook *Elliptic Partial Differential Equations of Second Order* by Gilbarg and Trudinger.

- The boundary conditions are not included in $w(x)$, thus $w(x)$ is only one possible solution.
- If $w(x)$ is well-defined, then it vanishes as $x \to \infty$.
- In a physical view, $w(x)$ is the Coulomb potential generated by the charge distributed as $f(x)$.

Some simple assumptions on $f(x)$ may produce that the Newtonian potential $w(x)$ is well-defined.

Theorem 5.2 (Regularity of Newtonian potential) *The Newtonian potential $w(x)$ satisfies*

- *If f is bounded and integrable in Ω, then $w \in C^1(\mathbb{R}^n)$ and*

$$D_i w(x) = \int_\Omega D_i \Gamma(x-y) f(y) \mathrm{d}y \tag{5.5}$$

- *If f is bounded and locally Hölder continuous (with exponent $\alpha \leqslant 1$), then $w \in C^2(\mathbb{R}^n)$ and $\Delta w = f$ in Ω, and*

$$D_{ij} w(x) = \int_{\Omega_0} D_{ij}\Gamma(x-y)(f(y)-f(x))\mathrm{d}y - f(x) \int_{\partial\Omega_0} D_i \Gamma(x-y) n_j(y) \mathrm{d}S_y \tag{5.6}$$

The RHS of (5.6) is calculated from

$$\begin{aligned} D_i w(x) &= \int_{\Omega_0} D_{ij}\Gamma(x-y) f(y) \mathrm{d}y \\ &= \int_{\Omega_0} D_{ij}\Gamma(x-y)(f(y)-f(x))\mathrm{d}y + f(x)\int_{\Omega_0} D_{ij}\Gamma(x-y)\mathrm{d}y \\ &= \int_{\Omega_0} D_{ij}\Gamma(x-y)(f(y)-f(x))\mathrm{d}y - f(x)\int_{\partial\Omega_0} D_i\Gamma(x-y) n_j(y)\mathrm{d}S_y \end{aligned}$$

The integral in (5.6) is well-defined because the integral

$$\int_{\Omega_0} |D_{ij}\Gamma(x-y)||x-y|^\alpha \mathrm{d}y \tag{5.7}$$

is well-defined for $\alpha > 0$. The theorem implies that we can derive the regularity of w from very weak assumptions on f.

Remark It is crucial to observe that although $w(x)$ is well-defined in $\mathbb{R}^n$, $w(x)$ is C^2 only in the region Ω. In fact, for $x \in \partial\Omega$, $D^2 w$ may not be defined.

Finally, we note that bounds of the derivatives $w(x)$ can be written. Suppose Ω is contained in the ball $B_R(x_0)$, where R is the radius. Then (5.5) implies

$$\begin{aligned}|D_i w(x)| &\leqslant \int_\Omega |D_i\Gamma(x-y)||f(y)|\mathrm{d}y \\ &\leqslant |f|_{0;\Omega}\int_\Omega |D_i\Gamma(x-y)|\mathrm{d}y \\ &\leqslant |f|_{0;\Omega}\int_{B_{2R}} |D_i\Gamma(x)|\mathrm{d}x \\ &\leqslant \frac{|f|_{0;\Omega}}{n\omega_n}\int_{B_{2R}} |x|^{1-n}\mathrm{d}x \\ &= \frac{|f|_{0;\Omega}}{n\omega_n}\cdot\omega_n\int_0^{2R}\mathrm{d}r \\ &= \frac{2R}{n}|f|_{0;\Omega}\end{aligned}$$

and (5.6) implies

$$\begin{aligned}|D_{ij}w(x)| &\leqslant \frac{|f(x)|}{n\omega_n}R^{1-n}\int_{B_{2R}}\mathrm{d}S_y + \frac{[f]_{\alpha;\Omega}}{\omega_n}\int_{B_2}|x-y|^{\alpha-n}\mathrm{d}y \\ &\leqslant 2^{n-1}|f(x)| + \frac{n}{\alpha}(3R)^\alpha[f]_{\alpha;\Omega}\end{aligned}$$

Theorem 5.3 (Estimation of Newtonian derivatives) *Let $w(x)$ be the Newtonian potential corresponding to $f(x)$, where $f(x)$ defined in Ω is bounded and local Hölder continuous with exponent α. Then the derivatives are approximated as*

$$\begin{aligned}|D_i w(x)| &\leqslant \frac{2R}{n}|f|_{0;\Omega} \\ |D_{ij}w(x)| &\leqslant 2^{n-1}|f(x)| + \frac{n}{\alpha}(3R)^\alpha[f]_{\alpha;\Omega}\end{aligned}\qquad x\in\Omega \tag{5.8}$$

Note that these inequalities are both scaling invariant. From the results above we obtain the following theorem.

Theorem 5.4 (Unique solution of the Poisson equation) *Let Ω be an bounded domain and suppose that each point of $\partial\Omega$ is regular. If f is bounded and Hölder continuous in Ω, then the classical Dirichlet problem $\Delta u = f$ is uniquely solvable with continuous boundary conditions.*

5.2 Hölder estimates of second-order derivatives

In the first part we have defined the Newtonian potential $w(x)$ corresponding to $f(x)$, i.e.,

$$w(x) = \int_\Omega \Gamma(x-y)f(y)\mathrm{d}y \tag{5.9}$$

We have shown that, if f is bounded and local Hölder continuous, then $w\in C^2(\Omega)$ and satisfies the classical equation $\Delta w = f$. In this part we consider more accurate bounds for the α-continuity of the solution.

Theorem 5.5 (continuity of Newtonian potential) *Let $B_1 = B_R(x_0)$ and $B_2 = B_{2R}(x_0)$ be concentric balls in $\mathbb{R}^n$. Suppose $f \in C^\alpha(\bar{B}_2)$, $0 < \alpha < 1$, and let w be the Newtonian potential of f in B_2, then $w \in C^{2,\alpha}(\bar{B}_1)$ and*

$$|D^2 w|'_{0,\alpha;B_1} \leqslant C|f|'_{0,\alpha;B_2} \tag{5.10}$$

or equivalently,

$$|D^2 w|_{0;B_1} + R^\alpha [D^2 w]_{\alpha;B_1} \leqslant C(|f|_{0;B_2} + R^\alpha [f]_{\alpha;B_2}) \tag{5.11}$$

where $C = C(n, \alpha)$.

The inequality is scaling invariant, since C only depends on n and α. Although it has been shown that $w \in C^2(B_2)$, the boundness of its derivatives can only be done in $\bar{B}_1$. This is why it is called an **interior estimate**. The proof of this theorem is done by directly estimating the difference $D^2 w(x_1) - D^2 w(x_2)$ using the expression of the second-order derivatives (5.6). The result above directly implies

Theorem 5.6 (solution regularity: compact support) *Let $u \in C_0^2(\mathbb{R}^n)$ and $f \in C_0^\alpha(\mathbb{R}^n)$ satisfy the Poisson's equation $\Delta u = f$ in $\mathbb{R}^n$. Then $u \in C_0^{2,\alpha}(\mathbb{R}^n)$ and if $B = B_R(x_0)$ is any ball containing the support of u, we have*

$$|D^2 u|'_{0,\alpha;B} \leqslant C|f|'_{0,\alpha;B}, \quad C = C(n, \alpha)$$
$$|u|'_{1;B} \leqslant CR^2 |f|_{0;B}, \quad C = C(n)$$

The first inequality is about the regularity of $D^2 w$, and the second inequality is about the derivatives of lower orders. The proof is based on the fact that $u(x)$ is exactly the Newtonian potential

$$u(x) = \int_{\mathbb{R}^n} \Gamma(x - y) f(y) \mathrm{d}y \tag{5.12}$$

While $D_i u$ can be directly bounded by (5.8), the bound of u itself is obtained from the integration from the boundary.

The above estimation of the Newtonian potential gives the following Schauder estimate.

Theorem 5.7 (Schauder interior estimate) *Let Ω be a domain in $\mathbb{R}^n$ and let $u \in C^2(\Omega)$ and $f \in C^\alpha(\Omega)$, satisfy the Poinsson's equation $\Delta u = f$. Then $u \in C^{2,\alpha}(\Omega)$ for any two concentric balls $B_1 = B_R(x_0)$ and $B_2 = B_{2R}(x_0) \subset\subset \Omega$, we have*

$$|u|'_{2,\alpha;B_1} \leqslant C(|u|_{0;B_2} + R^2 |f|'_{0,\alpha;B_2}) \tag{5.13}$$

Some important notations:

- The inequality is scaling invariant. The constant C does not depend on u or R or Ω.
- u is α-Hölder continuous in the whole region Ω, but its derivatives are only approximated in a smaller ball B_R.
- The theorem also holds true if the condition is weakened as $u \in C^2(\bar{B}_2)$ and $f \in C^\alpha(\bar{B}_2)$.

Proof The proof is based on the fact that any solution to the Poisson equation can be written as the sum of the Newtonian potential and a harmonic function. That is,

$$u(x) = v(x) + w(x), \quad x \in B_2 \tag{5.14}$$

where $v(x)$ is harmonic in B_2 and w is the Newtonian potential corresponding to f, that is,

$$w(x) = \int_{B_2} \Gamma(x-y) f(y)\mathrm{d}y, \quad x \in B_2 \tag{5.15}$$

(To apply the continuity theorem of the Newtonian potential, we must restrict our domain to B_2, a ball rather than the whole region Ω!) For $w(x)$, by applying the continuity theorem we have

$$R|Dw|_{0;B_1} + R^2|D^2w|'_{0,\alpha;B_1} \leqslant CR^2|f|'_{0,\alpha;B_2} \tag{5.16}$$

where the estimation of Dw is deduced from $|Dw|_{0;B_1} \leqslant CR|f|_{0;B_2}$. For the harmonic $v(x)$, its derivatives are bounded by the function value itself, thus

$$R|Dv|_{0;B_1} + R^2|D^2v|_{0,\alpha;B_1} \leqslant C|v|_{0;B_2} \tag{5.17}$$

Again use $v = u - w$. When $n > 2$, the function value of w can be approxiamted by

$$|w(x)| \leqslant |f|_{0;B_2} \int_{B_2} |\Gamma(x-y)|\mathrm{d}y \leqslant CR^2|f|_{0;B_2} \tag{5.18}$$

and thus

$$|v|_{0;B_2} \leqslant C(|u|_{0;B_2} + R^2|f|_{0;B_2}) \tag{5.19}$$

When $n = 2$, the proof is derived from $u(x_1, x_2) = u(x_1, x_2, x_3)$. Finally we obtain the result desired.

The result above implies the solutions of the Poisson equation are equicontinuous. By the Ascoli lemma,

Theorem 5.8 *Any bounded sequence of the Poisson equation $\Delta u = f$ with in Ω with $f \in C^\alpha(\Omega)$ contains a subsequence uniformly on compact subdomains of Ω.*

The Schauder interior estimate can be stated in alternative ways. Given the region Ω and $x, y \in \Omega$, define

$$d_x = \mathrm{dist}(x, \partial\Omega), \quad d_{x,y} = \min(d_x, d_y)$$

and the weighted norms

$$[u]^*_{k,0;\Omega} = [u]^*_{k,\Omega} = \sup_{x\in\Omega, |\beta|=k} d_x^k |D^\beta u(x)|$$

$$|u|^*_{k,0;\Omega} = |u|^*_{k,0;\Omega} = \sum_{j=0}^{k} [u]^*_{j;\Omega}$$

$$[u]^*_{k,\alpha;\Omega} = \sup_{x,y\in\Omega, |\beta|=k} d_{x,y}^{k+\alpha} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x-y|^\alpha}$$

$$|u|^*_{k,\alpha;\Omega} = |u|^*_{k,0;\Omega} + [u]^*_{k,\alpha;\Omega}$$

We also introduce the quantity

$$|f|^{(k)}_{0,\alpha;\Omega} = \sup_{x\in\Omega} d_x^k |f(x)| + \sup_{x,y\in\Omega} d_{x,y}^{k+\alpha} \frac{|f(x)-f(y)|}{|x-y|^\alpha} \tag{5.20}$$

The Schauder interior estimate can now be stated as

Theorem 5.9 (Schauder interior estimate) *Let $u \in C^2(\Omega)$ and $f \in C^\alpha(\Omega)$ satisfies $\Delta u = f$ in an open set Ω of $\mathbb{R}^n$, then*

$$|u|^*_{2,\alpha;\Omega} \leqslant C(|u|_{0;\Omega} + |f|^{(2)}_{0,\alpha;\Omega}) \tag{5.21}$$

where $C = C(n, \alpha)$.

The proof is based on the interior estimate with the radius $R = \frac{1}{3} d_x$ at given $x \in \Omega$.

It is important to note that the theorem contains the result of Theorem 5.7, a raw version of the Schauder interior estimate. By choosing the region Ω directly equal to the ball $B_2 = B_{2R}(x_0)$, we obtain

$$C(|u|_{0;B_2} + |f|^{(2)}_{0,\alpha;B_2}) \leqslant C(|u|_{0;B_2} + R^2|f|'_{0,\alpha;B_2}) \tag{5.22}$$

and

$$\begin{aligned}
|u|'_{2,\alpha;B_1} &= |u|_{0;B_1} + R|Du|_{0;B_1} + R^2|D^2u|_{0;B_1} + R^{2+\alpha}[D^2u]_{\alpha;B_1} \\
&= \sup_{x\in B_1} |u(x)| + R\sup_{x\in B_1} |Du(x)| + R^2 \sup_{x\in B_1} |D^2u(x)| + R^{2+\alpha} \sup_{x,y\in B_1} \frac{|D^2u(x) - D^2u(y)|}{|x-y|^\alpha} \\
&\leqslant \sup_{x\in B_1} |u(x)| + \sup_{x\in B_1} d_x|Du(x)| + \sup_{x\in B_1} d_x^2|D^2u(x)| + \sup_{x,y\in B_1} d_{x,y}^{2+\alpha} \frac{|D^2u(x) - D^2u(y)|}{|x-y|^\alpha} \\
&\leqslant \sup_{x\in B_2} |u(x)| + \sup_{x\in B_2} d_x|Du(x)| + \sup_{x\in B_2} d_x^2|D^2u(x)| + \sup_{x,y\in B_2} d_{x,y}^{2+\alpha} \frac{|D^2u(x) - D^2u(y)|}{|x-y|^\alpha} \\
&= |u|^*_{2,\alpha;B_2}
\end{aligned}$$

(note that d_x is the distance between $x \in B_1$ and ∂B_2, and we must have $d_x \geqslant R$ and $d_{x,y} \geqslant R$.) Therefore we obtain the Schauder estimate for two concentric balls.

5.3 Estimations on the boundary

In order to establish the global estimate, we need to derive the Schauder estimate on the boundary. Let $\mathbb{R}^+_n$ be the upper half space $\{x_n > 0\}$, and T be the hyperplane $\{x_n = 0\}$. $B_1 = B_R(x_0)$ and $B_2 = B_{2R}(x_0)$ as before. Let $B_i^+ = B_i \cap \mathbb{R}^n_+$.

Theorem 5.10 (continuity of Newtononian potential) *Let $u \in C^\alpha(\bar{B}_2^+)$ and let w be the Newtonian potential of f in B_2^+. Then $w \in C^{2,\alpha}(\bar{B}_1^+)$ and*

$$|D^2w|'_{0,\alpha;B_1^+} \leqslant C(n,\alpha)|f|'_{0,\alpha;B_2^+} \tag{5.23}$$

where $C = C(n, \alpha)$.

This result is scaling invariant. The constant C does not depend on the radius R, and even how much $\bar{B}_2^+$ interacts with its boundary. The proof is based on the fact that when either i or j is not n, the second-order derivative of the Newtonian potential $w(x)$ can be written as

$$D_{ij}w(x) = \int_{B_2^+} D_{ij}\Gamma(x-y)(f(y)-f(x))\mathrm{d}y - f(x)\int_{\partial B_2^+} D_i\Gamma(x-y)n_j(y)\mathrm{d}S_y \tag{5.24}$$

and the integral on the boundary ∂B_2 vanishes on T. Thus we can derive the continuity of $D_{ij}w(x)$ except D_{nn}, and

$$|D_{ij}w|'_{0,\alpha;B_1^+} \leqslant C_{ij}(n,\alpha)|f|'_{0,\alpha;B_2^+} \tag{5.25}$$

Finally, the estimation of $D_{nn}w$ is derived from $\Delta w = f$, which implies

$$D_{nn}w(x) = f(x) - \sum_{i=1}^{n-1} D_{ii}w(x) \tag{5.26}$$

Now we can derive the Schauder estimate on the boundary.

Theorem 5.11 (Schauder estimate: boundary) *Let $u \in C^2(B_2^+) \cap C(\bar{B}_2^+)$ and $f \in C^\alpha(\bar{B}_2^+)$ satisfy $\Delta u = f$ in B_2^+ and $u = 0$ on T. Then $u \in C^{2,\alpha}(\bar{B}_1^+)$ and*

$$|u|'_{2,\alpha;B_1^+} \leqslant C(|u|_{0;B_2^+} + R^2|f|'_{0,\alpha;B_2^+}) \tag{5.27}$$

where $C = C(n,\alpha)$.

Proof Let $x^* = (x', -x_n)$ be the reflecting point of $x \in \mathbb{R}^n$. The crucial step in this theorem is to construct a Newtonian-like potential function

$$w(x) = \int_{B_2^+} [\Gamma(x-y) - \Gamma(x^*-y)]f(y)\mathrm{d}y \tag{5.28}$$

which satisfies

- $w(x) = 0$ for $x_n = 0$;
- $\Delta w(x) = f(x)$ in the upper half ball B_2^+.

Since $w(x)$ can be written as

$$w(x) = 2\int_{B_2^+} \Gamma(x-y)f(y)\mathrm{d}y - \int_D \Gamma(x-y)f^*(y)\mathrm{d}y \tag{5.29}$$

By applying the continuity theorem on B_2^+ and D, we conclude that $w \in C^\alpha(\bar{B}_2^+)$. Thus the difference $v = u - w$ satisifies

- $v(x) = 0$ in T;
- $v(x)$ is harmonic in B_2^+.

Since the derivatives of $v(x)$ can be approximated, the result holds for a general solution u. Similarly we obtain

Theorem 5.12 (Schauder interior estimate: boundary) *Let Ω be an open subset of in $\mathbb{R}^n_+$ with a boundary portion T on $x_n = 0$. Let $u \in C^2(\Omega) \cap C(\Omega \cup T)$ be the solution to $\Delta u = f$, where $f \in C^\alpha(\Omega)$, and $u = 0$ on T, then*

$$|u|^*_{2,\alpha;\Omega\cup T} \leqslant C(\||u|_{0;\Omega}\| + \|f\|^{(2)}_{0,\alpha;\Omega}) \tag{5.30}$$

where $C = C(n, \alpha)$.

6 Schauder estimate: [CW]

6.1 Existence of the solution

Theorem 6.1 (maximum principle) *Consider the elliptic operator L given by*

$$Lu = -a^{ij} D_{ij} u + b^i D_i u + cu \tag{6.1}$$

satisfying

- $\lambda|\xi|^2 \leqslant a^{ij}(x)\xi_i\xi_j \leqslant \Lambda|\xi|^2$, $x \in \Omega$, $\xi \in \mathbb{R}^n$;
- $|b(x)| \leqslant M$, $x \in \Omega$;
- $c(x) \geqslant 0$, $x \in \Omega$

If $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfies $Lu \leqslant f$ in Ω, then

$$\sup_\Omega u \leqslant \sup_{\partial\Omega} u^+ + C|f|_{0;\Omega} \tag{6.2}$$

for some constant C.

This theorem is exactly the maximum principle for the second-order elliptic equation.

(1) First assume there is a constant c_0 such that $c(x) \geqslant c_0$. Translate the function $u(x)$ by

$$v(x) = u(x) - \sup_{\partial\Omega} u^+, \qquad x \in \bar{\Omega} \tag{6.3}$$

then v satisfies

$$\begin{aligned} Lv &= f - c\sup_{\partial\Omega} u^+ \leqslant f, \quad & x \in \Omega \\ v &\leqslant 0, & x \in \partial\Omega \end{aligned} \tag{6.4}$$

Suppose v attains maximum at $x_0 \in \Omega$, then

$$D^2 v(x_0) \leqslant 0, \qquad Dv(x_0) = 0 \tag{6.5}$$

From $Lv(x_0) \leqslant f(x_0)$ we obtain

$$c(x_0)v(x_0) \leqslant f(x_0) \leqslant |f|_{0;\Omega} \tag{6.6}$$

which implies $v(x_0) \leqslant \frac{1}{c_0}|f|_{0;\Omega}$. Recall that $v(x_0)$ is the maximum attained by v, we have

$$\sup_{x\in\Omega} v(x) \leqslant \frac{|f|_{0;\Omega}}{c_0} \tag{6.7}$$

which ends the proof.

(2) For the general case, we set $v(x) = z(x)w(x)$, where $z(x)$ is a given function. Then $w(x)$ satisfies the second-order equation

$$-a^{ij}D_{ij}w + \left(b^i - \frac{2}{z}a^{ij}\frac{\partial z}{\partial x_i}\right)D_i w + \left[c + \frac{1}{z}(b^i D_i z - a^{ij}D_{ij}z)\right]w \leqslant \frac{f}{z} \tag{6.8}$$

When Ω is contained in the region $0 < x_1 < d$, $z(x)$ can be chosen as

$$z(x) = e^{2\alpha d} - e^{\alpha x_1} \tag{6.9}$$

where $\alpha > 0$ is a given constant.

7 Schauder estimate: Paper

Let's consider the Poisson equation

$$\Delta u = f, \qquad x \in B_1(0) \tag{7.1}$$

Suppose f is Dini continuous, namely

$$\int_0^1 \frac{w(r)}{r}\mathrm{d}r < +\infty, \qquad w(r) := \sup_{|x-y|<r} |f(x) - f(y)| \tag{7.2}$$

then we have the following estimation of D^2u, if u is a C^2 solution to the equation (7.1).

Theorem 7.1 (Schauder estimate) *Let $u \in C^2(B_1)$ be a classical solution to the Laplace equation (7.1). Then for $x, y \in B_{\frac{1}{2}}(0)$,*

$$|D^2u(x) - D^2u(y)| \leqslant C_n\left[d\sup_{B_1}|u| + \int_0^d \frac{w(r)}{r}\mathrm{d}r + d\int_d^1 \frac{w(r)}{r^2}\right] \tag{7.3}$$

where $d = |x-y|$, C_n only depends on n.

Remark $w(r)$ is an increasing function of r. When f is α-Hölder continuous, it can be shown that

$$\int_0^d \frac{w(r)}{r}\mathrm{d}r + d\int_d^1 \frac{w(r)}{r^2} = O(d^\alpha) \tag{7.4}$$

which implies $|D^2u(x) - D^2u(y)| \leqslant Cd^\alpha$, and D^2u is thus Hölder continuous.

Proof

1. Definition of the function sequence u_k shrinking to x_0

Let $\rho = \frac{1}{2}$ be a constant. Fix some point $x_0 \in B_{\frac{1}{2}}(0)$ and define the family of balls centered at x_0, defined by

$$B_k^{(0)} = \{x \in \mathbb{R}^n : |x - x_0| \leqslant \rho^k\}, \qquad k \geqslant 1 \tag{7.5}$$

and $B_0^{(0)} = B_1(0)$. It's easy to see $B_k^{(0)} \subset B_1(0)$ for any $k \geqslant 0$, and $B_k^{(0)}$ shrinks to 0 as $k \to \infty$. Now let u_k be the classical solution to the Poisson equation with Dirichlet boundary conditions:

$$\begin{cases} \Delta u_k = f(x_0), & x \in B_k^{(0)} \\ u_k = u, & x \in \partial B_k^{(0)} \end{cases} \tag{7.6}$$

then $u_k \in C^2(B_k^{(0)})$ and satisfies

$$\Delta(u_k - u) = f(x_0) - f, \quad x \in B_k^{(0)} \tag{7.7}$$

For $x \in B_k^{(0)}$, we always have $|f(x_0) - f(x)| \leqslant \rho^k$. Note that $u_k - u$ also vanishes at the boundary $\partial B_k^{(0)}$, hence from the maximum principle,

$$\|u_k - u\|_{L^\infty(B_k^{(0)})} \leqslant C\rho^{2k}\omega(\rho^k), \quad k \geqslant 0 \tag{7.8}$$

Differencing (7.8) at k and $k+1$, we obtain

$$\|u_k - u_{k+1}\|_{L^\infty(B_{k+1}^{(0)})} \leqslant C\rho^{2k}w(\rho^k), \quad k \geqslant 0 \tag{7.9}$$

Since $u_k - u_{k+1}$ is harmonic in $B_{k+2}^{(0)}$ (whose radius is $O(\rho^k)$), the derivatives of $u_k - u_{k+1}$ in the ball $B_{k+2}^{(0)}$ can be approximated as

$$\|D(u_k - u_{k+1})\|_{L^\infty(B_{k+2}^{(0)})} \leqslant C\rho^k w(\rho^k) \tag{7.10}$$

$$\left\|D^2(u_k - u_{k+1})\right\|_{L^\infty(B_{k+2}^{(0)})} \leqslant Cw(\rho^k) \tag{7.11}$$

2. u_k characterizes the local differentiability of u at x_0

Since $u \in C^2(B_1)$, let $q(x)$ be the quadratic part of u localized at x_0, i.e.,

$$q(x) = u(x_0) + Du \cdot (x - x_0) + \frac{1}{2}(x - x_0)^{\mathrm{T}} D^2u(x_0)(x - x_0), \quad x \in B_1 \tag{7.12}$$

then $q(x)$ has the same local properties with $u(x)$ at x_0, i.e.,

$$Dq(x_0) = Du(x_0), \quad D^2q(x_0) = D^2u(x_0) \tag{7.13}$$

and $u_k - q$ is thus harmonic in $B_k^{(0)}$. Also note that from the Taylor expansion,

$$u(x) - q(x) = o(|x - x_0|^2), \quad x \to x_0 \tag{7.14}$$

hence there exists a sequence $\varepsilon_k \to 0$ such that

$$\|u - q\|_{L^\infty(B_k^{(0)})} \leqslant \varepsilon_k \rho^{2k}, \quad k \geqslant 0 \tag{7.15}$$

Using (7.8)(7.15) we obtain there exists a sequence $\varepsilon_k \to 0$ such that

$$\|u_k - q\|_{L^\infty(B_k^{(0)})} \leqslant \varepsilon_k \rho^{2k}, \quad \forall k \geqslant 0 \tag{7.16}$$

Using the fact that $u_k - q$ is harmonic, we obatin the derivative estimate

$$|Du_k(x_0) - Du(x_0)| \leqslant C\varepsilon_k \rho^k \tag{7.17}$$

$$|D^2u_k(x_0) - D^2u(x_0)| \leqslant C\varepsilon_k \tag{7.18}$$

Let $k \to \infty$, we finally obtain

$$\lim_{k\to\infty} Du_k(x_0) = Du(x_0), \qquad \lim_{k\to\infty} D^2u_k(x_0) = D^2u(x_0) \tag{7.19}$$

3. Estimate $D^2u(z) - D^2u(x_0)$ near x_0

Now let's estimate $D^2u(z) - D^2u(x_0)$ when the space variable z is near x_0. Write

$$|D^2u(z) - D^2u(x_0)| \leqslant I_1 + I_2 + I_3 \tag{7.20}$$

where I_1, I_2, I_3 are given by

$$\begin{aligned} I_1 &= |D^2u_k(z) - D^2u_k(x_0)| \\ I_2 &= |D^2u_k(x_0) - D^2u(x_0)| \\ I_3 &= |D^2u_k(z) - D^2u(z)| \end{aligned}$$

Suppose there exists $k \geqslant 1$ such that $\rho^{k+4} \leqslant |z-x_0| \leqslant \rho^{k+3}$. Now we estimate I_1, I_2, I_3 respectively.

- I_1 is about the local continuity of $D^2u_k(x)$ at x_0. For $j = 1, \cdots, k-1$, define

  $$h_j(x) = u_{j+1}(x) - u_j(x), \qquad x \in B_k^{(0)} \tag{7.21}$$

  then (7.11) implies that

  $$\left\|D^2h_j\right\|_{L^\infty(B_{k+2}^{(0)})} \leqslant Cw(\rho^k), \quad j = 1, \cdots, k-1 \tag{7.22}$$

  Note that $z \in B_{k+2}^{(0)}$ and $|z - x_0| = O(\rho^k)$, thus (7.22) implies

  $$|D^2h_j(z) - D^2h_j(x_0)| \leqslant C\rho^{-j}w(\rho^j)|z - x_0| \tag{7.23}$$

  Now use

  $$D^2u(z) - D^2u(x_0) = (D^2u_0(z) - D^2u_0(x_0)) + \sum_{j=0}^{k-1}(D^2h_j(z) - D^2h_j(0)) \tag{7.24}$$

  We have

  $$\begin{aligned} |D^2u_k(z) - D^2u_k(x_0)| &\leqslant |D^2u_0(z) - D^2u_0(x_0)| + \sum_{j=0}^{k-1}|D^2h_j(z) - D^2h_j(0)| \\ &\leqslant |D^2u_0(z) - D^2u_0(x_0)| + C|z - x_0|\sum_{j=0}^{k-1}\rho^{-j}w(\rho^j) \\ &\leqslant |D^2u_0(z) - D^2u_0(x_0)| + C|z - x_0|\int_{|z-x_0|}^1 \frac{w(r)}{r^2}\mathrm{d}r \end{aligned}$$

Therefore, our problem reduces to the approximation of $|D^2u_0(z) - D^2u_0(x_0)|$. Again define the quadratic part of $u_0(x)$ as

$$q_0(x) = u_0(x_0) + Du_0 \cdot (x - x_0) + \frac{1}{2}(x - x_0)^{\mathrm{T}} D^2u_0(x - x_0), \quad x \in B_1(0) \tag{7.25}$$

then $D^2q_0(x_0) = D^2u_0(x_0)$, $u_0 - q_0$ is harmonic. Note that u_0 is a classical solution to the Poisson equation, we have

$$u_0(x) - q_0(x) = O(|x - x_0|^3), \quad x \in B_1(0) \tag{7.26}$$

hence there exists a constant C such that

$$|D^2u_0(z) - D^2u_0(x_0)| \leqslant C|z - x_0| \|u_0\|_{L^\infty}, \quad x \in B_1(0) \tag{7.27}$$

Using the approximation (7.8), we obtain

$$|D^2u_0(z) - D^2u_0(x_0)| \leqslant C|z - x_0| \Big(\|u\|_{L^\infty} + Cw(\rho) \Big) \tag{7.28}$$

Finally, we obtain

$$I_1 \leqslant C|z - x_0| \left(\|u\|_{L^\infty} + \int_{|x-x_0|}^1 \frac{w(r)}{r^2} \mathrm{d}r \right) \tag{7.29}$$

- I_2 is about the difference between D^2u_k and D^2u. Using (7.11) we have

$$\left\| D^2h_j \right\|_{L^\infty(B_{j+2}^{(0)})} \leqslant Cw(\rho^j), \quad \forall j \geqslant k \tag{7.30}$$

which implies

$$|D^2u_j(x_0) - D^2u_{j+1}(x_0)| \leqslant Cw(\rho^j), \quad \forall j \geqslant k \tag{7.31}$$

Sum this result over $j = k+1, k+2, \cdots$, we have

$$|D^2u_k(x_0) - D^2u(x_0)| \leqslant C \sum_{j=k}^{\infty} w(\rho^k) \leqslant C \int_0^{|x-x_0|} \frac{w(r)}{r} \mathrm{d}r \tag{7.32}$$

which implies

$$I_2 \leqslant C \int_0^{|x-x_0|} \frac{w(r)}{r} \mathrm{d}r \tag{7.33}$$

- The estimation of I_3 is similar with I_2, by considering the balls

$$B_j^{(1)} := \{x \in \mathbb{R}^n : |x - z| \leqslant \rho^j\}, \quad j \geqslant k+1 \tag{7.34}$$

instead of $B_j^{(0)}$. Correspondingly, let $v_j(x)$ $(j \geqslant k+1)$ be the solution to

$$\begin{cases} \Delta v_j = f(z), & x \in B_j^{(1)} \\ v_j = u, & x \in \partial B_j^{(1)} \end{cases} \tag{7.35}$$

and $v_j(x) = u_j(x)$, then similarly we have

$$|D^2 v_j(z) - D^2 v_{j+1}(z)| \leqslant C w(\rho^j), \quad \forall j \geqslant k \tag{7.36}$$

which implies

$$I_3 \leqslant C \int_0^{|x-x_0|} \frac{w(r)}{r} \mathrm{d}r \tag{7.37}$$

Finally obtain from the estimation of I_1, I_2, I_3 that

$$|D^2 u(z) - D^2 u(x_0)| \leqslant C \left[d \sup_{B_1} |u| + \int_0^d \frac{w(r)}{r} \mathrm{d}r + d \int_d^1 \frac{w(r)}{r^2} \right] \tag{7.38}$$

with $d = |z - x_0|$ is the distance.

8 L^p estimate

Definition 8.1 (Distribution function) *Suppose $f \in L^1(\Omega)$ and define the set*

$$A(t) := \{x \in \Omega : |f(x)| > t\}$$

The function $\lambda(t) := \mathrm{meas}(A(t))$ is called the distribution function f.

$\lambda(t)$ characterizes how large is the region that $|f(x)| > t$ in it. Now L^p-integral of f can be easily expressed via the integral of $\lambda(t)$.

$$\int_\Omega |f(x)|^p \mathrm{d}x = p \int_0^\infty t^{p-1} \lambda(t) \mathrm{d}t$$

Now we define the Marcinkiewicz space.

Definition 8.2 (Marcinkiewicz space) *For $p \geqslant 1$,*

$$\|f\|_{L^p_w} := \left(\sup_{t \geqslant 0} \{t^p \lambda(t)\} \right)^{\frac{1}{p}}$$

For any $t \geqslant 0$, we have

$$t^p \lambda(t) \leqslant \int_{A(t)} |f(x)|^p \mathrm{d}x \leqslant \int_\Omega |f(x)|^p \mathrm{d}x$$

Taking the supremum of t we have

$$\sup_{t \geqslant 0} \{t^p \lambda(t)\} \leqslant \int_\Omega |f(x)|^p \mathrm{d}x$$

which implies

$$\|f\|_{L^p_w} \leqslant \|f\|_{L^p}$$

thus $\|\cdot\|_{L^p_w}$ is weaker than the standard L^p norm. Now we prove that for any $q < p$, $L^p_w(\Omega) \subset L^q(\Omega)$. In fact,

$$\begin{aligned}\int_\Omega |f|^q \mathrm{d}x &= q\int_0^\infty t^{q-1}\lambda(t)\mathrm{d}t \\ &= q\int_0^1 t^{q-1}\lambda(t)\mathrm{d}t + \int_1^\infty qt^{q-1}\lambda(t)\mathrm{d}t \\ &\leqslant q|\Omega| + q\cdot\sup_{t\geqslant 0}\{t^p\lambda(t)\}\int_1^\infty t^{q-p-1}\mathrm{d}t \\ &< \infty\end{aligned}$$

thus for $1 \leqslant q < p$ we have

$$L^p(\Omega) \subset L^p_w(\Omega) \subset L^q(\Omega)$$

Characterization of the L^p functions:

- $L^p(\Omega)$: $p\int_\Omega t^{p-1}\lambda(t)\mathrm{d}x < +\infty$ (stronger)
- $L^p_w(\Omega)$: $\sup_{t\geqslant 0}\{t^p\lambda(t)\} < +\infty$ (weaker)